THE OBSTRUCTION TO THE FINITENESS OF THE TOTAL SPACE OF A FIBRATION

Karel Ehrlich

INTRODUCTION

Given a space E, we say it is finitely dominated if it is a homotopy retract of a finite CW complex. That is, there is a finite CW complex K and maps $E \to K \to E$ so that the composition is homotopic to 1_E. Wall [20] showed that in that case the singular chain complex of the universal cover $\tilde{E}$ is chain equivalent to a finite, finitely generated projective chain complex P_i over the group ring $\mathbf{Z}\pi_1 E$. He further showed that $O(E) = \sum (-1)^i P_i \in \tilde{K}_0(\pi_1 E)$ is zero if and only if E has the homotopy type of a finite CW complex.

Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration, where both the fiber F and base B are finitely dominated. Then E is also finitely dominated [17] and the question arises: how is $O(E)$ related to $O(\mathrm{B})$ and $O(F)$? For the trivial fibration $E = F \times B$, there is the product formula of Siebenmann [19] and Gersten [13],

$$O(E) = \chi(F)\cdot s_* O(B) + \chi(B)\cdot i_* O(F) + O(B) \otimes O(F),$$

where $\chi(F)$ is the Euler characteristic of F, and s_*, i_* are maps from $\tilde{K}_0(\pi_1 B)$, $\tilde{K}_0(\pi_1 F)$ into $\tilde{K}_0(\pi_1 E)$ induced by the corresponding maps of the fundamental groups. Lal [17] obtained the same formula for a fibration in which the base B has the homotopy type of a finite complex. The product formula breaks down, however, if $O(B) \neq 0$. Anderson [6] computed $p_* O(E) \in \tilde{K}_0(\pi_1 B)$ for some fibrations and showed that the orientation of the fibration had to be taken into account. I showed that $p_* O(E)$ depends only on B and F and the orientation [10], [11]. Then Pedersen and Taylor [18] gave an explicit formula for $p_* O(E)$.

In this paper we are interested in the actual computations of $O(E) \in \tilde{K}_0(\pi_1 E)$ rather than its image in $\tilde{K}_0(\pi_1 B)$. The only known result in this direction is Anderson [5]. He shows that $O(E) = 0$ for a principal S^1 bundle, $S^1 \to E \to B$, where $\pi_1 S^1$ injects into abelian $\pi_1 E$. I derive several formulae and show that under various conditions an analog of the product formula holds for orientable fibrations.

By an orientable fibration I shall mean a Hurewicz fibration $F \to E \to B$, which is a pullback of a fibration with a simply connected base space. That means given a space F, the orientable fibrations with fiber F are classified by a universal fibration $F \to EF \to BF$, where BF is simply connected. This notion of orientability

Received April 7, 1978. Revision received February 8, 1979.
Second revision received June 7, 1979.

Michigan Math J. 28 (1981).