A TRULY ISOLATED UNIVALENT FUNCTION

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Let R denote the space of functions

$$f(z)=\sum_{n=1}^{\infty} a_n z^n \tag{1}$$

that are holomorphic in the unit disc D and let the space R be metrized by the metric

$$\|f\|=\sup_n |a_n|^{1/n}. \tag{2}$$

Let S denote the subset of R which consists of the functions $\Sigma a_n z^n$ that are univalent (but not necessarily normalized) in D. Hornich [2] studied the structure of the subset S. In [3] he exhibited a function f in S with the following property: for some positive number r none of the functions $f(z)+cz$ $(0<|c|<r$, c not positive) belongs to S. Hornich's example suggested that S may have isolated points. Piranian [5] claimed there is a function $f(z)=\Sigma a_n z^n$ which belongs to S and lies at a distance one from $S-\{f\}$. His proof relied heavily on certain geometric constructions for which it is difficult to obtain detailed proofs. The following proof is much simpler and the claim is much stronger. Instead of a suggestive geometric argument it utilizes known geometric properties of normal analytic functions together with an explicit analytic construction.

THEOREM. *Let LS denote the set of functions in R which are locally univalent in D. Then there exists a univalent function f(z) in R which lies at a distance one from LS* $-\{f\}$.

Thus not only is this univalent function isolated from other univalent functions, it is even isolated from all other locally univalent functions. The proof also provides a univalent function such that no function k in R with $\|k-f\|<1$ is univalent (or even locally univalent) in *any sector of the unit disc,* a result which Piranian observed would be at best tedious to derive from his geometrical approach [5, p. 238].

We first prove a lemma which provides a simple explicit construction of an analytic function without finite radial limits.

LEMMA 1. *Let* $h(z)=\sum_{n=1}^{\infty} z^{g(n)}$ *where* $g(n)=2^{2^n}$. *Then h(z) has no finite radial limits and furthermore* $(1-|z|)\,|h'(z)|\le 2$.

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