SOME EXTREMAL PROBLEMS IN CONFORMAL AND QUASICONFORMAL MAPPING

Andrew N. Harrington

INTRODUCTION

This paper centers around extremal problems for which the extremal functions are univalent mappings of multiply connected domains in the extended complex plane onto domains bounded by arcs of generalized lemniscates, arcs where

$$\prod_{j=1}^{n} |w - w_j|^{x_j}$$

is constant for some complex constants $w_1, w_2, \ldots, w_n$ and nonzero real numbers $x_1, x_2, \ldots, x_n$. We formulate an extremal problem for conformal mappings, apply variational methods, and prove that the solution of this extremal problem provides a new proof of a canonical mapping theorem. We modify the extremal problem so it applies to general classes of quasiconformal mappings in order to obtain a new representation of the dielectric Green's function for multiply connected domains.

The theory of lemniscates has always played a significant role in the theory of conformal mapping. On one hand, Hilbert has shown that one can approximate quite general sets of continua by lemniscates. On the other hand, the Green's function with pole at ∞ is obviously elementary in the case of a domain whose boundary is an entire lemniscate.

Julia [4, Chapter 5] extending the work of de la Vallée Poussin, proved

THEOREM 1. *To each n-tuply connected domain Δ in the complex plane there corresponds a polynomial $P(z) = \prod_{j=1}^{n-1} (z - z_j)$ such that Δ can be mapped conformally onto a domain with the property that $|P(z)|$ is constant on each component of the boundary.*

Walsh [21] proved theorems about conformal mappings onto regions bounded by all of one or two generalized lemniscates. Jenkins [3], Landau [7], and Pirl [10] have given alternate proofs. In [22] Walsh and Landau considered a limiting case of Walsh's original theorems in which different boundary continua come together. De la Vallée Poussin, Julia, Walsh, and Jenkins all relied on uniformization theorems for their proofs. Landau noted that the lemniscate mappings transform certain harmonic domain functions into functions which are extendible harmonically

Received July 6, 1978; Revision received November 1, 1978

Michigan Math. J. 27 (1980).