APPLICATIONS OF CONVOLUTION OPERATORS TO PROBLEMS IN UNIVALENT FUNCTION THEORY

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In this paper we investigate a wide class of problems. We will exploit the strengths and properties of convolution operators. The strength of these methods lies in their ability to unify a number of diverse results. Of the previously known results obtained in this paper, most of the earlier proofs were tedious examinations of the specific properties of the classes of functions involved. In this paper we are able to obtain and generalize many of these results and obtain a number of new results including a verification of Robinson's 1/2 conjecture in the case of spirallike functions. In general, the proofs using convolution operators are clearer and more concise and point out how the unifying linear structure that is common to so many of the problems can be used to solve them via convolution operator techniques.

PRELIMINARY RESULTS

The unit disk in the complex plane will be denoted by U. Let A be the class of analytic functions on U. Let S denote those functions in A that are univalent and normalized by $f(0) = 0$ and $f'(0) = 1$. Let C, S^*, K and S_p be the standard subclasses of S consisting of the convex, starlike, close-to-convex, and spirallike functions, respectively. Let P be the class of functions p in A which have positive real part and are normalized by $p(0) = 1$. Let K_1 be the class of function f in S that have f' in P.

If f and g are in A with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, the convolution of f and g is defined by $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. Given f in A, we define the convolution operator $\Gamma: A \to A$ by $\Gamma(g) = f * g$.

We will use the results and techniques of Ruscheweyh and Sheil-Small developed in [18] in connection with their proof of the Polya-Schöenberg conjecture. Specifically, the following theorem of theirs and the key lemma used in its proof will be used in our work.

THEOREM A. *Let h be in C. If f is in C, S^* or K then $h * f$ is in C, S^* or K respectively.*

In their proof of Theorem A, they proved a most interesting key lemma. We shall need the slightly more general version of their key lemma stated without proof in their paper. For completeness we include a proof of the more general

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