LIFTING OF A CONTRACTION INTERTWINING TWO ISOMETRIES

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0. Throughout this note we consider only (bounded, linear) operators on Hilbert spaces. As usual, we denote by $L(\mathcal{H}_1,\mathcal{H}_2)$ the space of all operators from $\mathcal{H}_1$ into $\mathcal{H}_2$ and by $L(\mathcal{H})$ the space $L(\mathcal{H},\mathcal{H})$. Also, for two contractions T_1 and T_2 on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, we shall denote by $I(T_2,T_1)$ the set of all operators $A \in L(\mathcal{H}_1,\mathcal{H}_2)$ intertwining T_2 and T_1; *i.e.*, satisfying $T_2A = AT_1$. Let $V_i \in L(\mathcal{H}_i)$ be an isometry ($i = 1,2$), $\mathcal{H}_0$ a (closed linear) subspace of $\mathcal{H}_1$ invariant for V_1 and $V_0 = V_1 | \mathcal{H}_0$. By a *contractive intertwining lifting in* $I(V_2, V_1)$ (briefly, (V_2, V_1)-CIL) of a contraction $A \in I(V_2, V_0)$ we mean any contraction $\hat{A} \in I(V_2, V_1)$ satisfying $\hat{A} | \mathcal{H}_0 = A$. In case $V_i = S_i$ ($i = 1,2$) is a unilateral shift, necessary and sufficient conditions for the existence and the uniqueness of such a (S_2,S_1)-CIL were given in [3, Theorem 2] and [4, Proposition 3.1] Also, in case $V_i = U_i$ ($i = 1,2$) is a unitary operator and $\mathcal{H}_0$ is a reducing subspace for U_1, three equivalent conditions for the uniqueness of a (U_2, U_1)-CIL of A (which obviously, in this case always exists) were given in [2, Corollary 2.3]. In the present note we extend the result of [3] to the case of arbitrary isometries V_1 and V_2 (see Thm. 1.1 below), and also, adapting the quoted results of [2] and [4], we give a necessary and sufficient condition for the uniqueness of (V_2, V_1)-CIL of A (see Section 3, Thm. 3.1).

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1. THEOREM 1.1. *Let $V_i \in L(\mathcal{H}_i)$ be an isometry ($i = 1,2$), let $\mathcal{H}_0$ be a subspace of $\mathcal{H}_1$ invariant for V_1, let $V_0 = V_1 | \mathcal{H}_0$, and let A be a contraction belonging to $I(V_2, V_0)$. Then, there exists a (V_2, V_1)-CIL $\hat{A}$ of A if and only if the condition*

$$\|(I - V_2^n V_2^{*n}) A h_0\| \leq \|(I - V_1^n V_1^{*n}) h_0\| \tag{1.1}$$

holds for all $n = 1,2,\dots$ and $h_0 \in \mathcal{H}_0$.

Proof. Since the necessity of the condition (1.1) is obvious, it remains to prove its sufficiency. For this purpose we adapt the original proof of [3] to the present more general situation.

Let $U_i \in L(\mathcal{K}_i)$ be the minimal unitary dilation of V_i ($i = 0,1,2$) (see [9, Ch.I, Sec. 4]); obviously we can and shall identify $\mathcal{K}_0$ with the space $\bigvee_{n=0}^{\infty} U_1^{-n} \mathcal{H}_0$ and $U_0 = U_1 | \mathcal{K}_0$. Also, let us denote by P_i the orthogonal projection of $\mathcal{K}_i$ onto $\mathcal{H}_i$ ($i = 0,1,2$), and let us set $\mathcal{G}_i = (I - P_i) \mathcal{K}_i$ ($i = 1,2$) and $\mathcal{G}_0 = ((I - P_1) \mathcal{K}_0)^-$.

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