# ON THE SINGULARITY SET OF COMPLEX FUNCTIONS SATISFYING THE CAUCHY-RIEMANN EQUATIONS

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## 1. INTRODUCTION

Let $f(z) = u(x,y) + iv(x,y)$ be a finite valued complex function defined on a domain $D$ and satisfying the Cauchy-Riemann equations everywhere in $D$; *i.e.* at every point $u$ and $v$ possess finite first order partials satisfying $u_x = v_y$, $u_y = -v_x$. Under the additional restriction that $f$ be continuous, or even only locally bounded, it is known that $f$ must be analytic in $D$: theorems of Looman-Menchoff [3], [5] and Tolstov [4] respectively. With no supplementary restriction $f$ need not be analytic everywhere in $D$ (consider *e.g.* $f(z) = e^{-1/z^4}$), but Trokhimchuk ([5, p. 109f]) proved that the singularity set $B$ is a closed totally disconnected set whose projections on the coordinate axes are closed nowhere dense linear sets (a result whose proof required Tolstov's theorem) and asked whether it was possible for $B$ to contain a (perfect) nucleus. It will be shown here that (section 2) $B$ can be non-denumerable and even of positive Lebesgue measure with $f$ satisfying certain additional imposed conditions. Some further questions are raised in section 3.

## 2. LARGE SINGULARITY SETS

*Definition.* A complex function $f(z) = u + iv$ has a *directional derivative* $f'(a;z)$ in the direction $a = e^{i\theta}$ at $z$ if $\lim_{h \to 0^+} (f(z + ah) - f(z))/ah$ exists finitely and equals $f'(a;z)$. In particular if $f'(\pm 1;z)$, $f'(\pm i;z)$ all exist and are equal then $f$ is said to satisfy *CR* at $z$ which is equivalent to $u, v$ having first order partials at $z$ obeying the Cauchy-Riemann equations.

LEMMA 1. *Let $\{a_1, a_2, \ldots\}$ be a countable set of directions, $|a_i| = 1$, and let $D$ be the unit disc $|z| \leq 1$. Then there exists a countable isolated subset $A = \{b_1, b_2, \ldots\}$ of $D$ and disjoint open discs $N_i$ centred on $b_i$, $i \geq 1$, such that if $p_i$ is the orthogonal projection on the tangent $L_i$ to $D$ at $a_i$ then for each $i \geq 1$ the sets $p_i \bar{N}_j \subseteq L_i$ are disjoint for $j \geq i$. Furthermore $A$ can be chosen so that $K = \bar{A} \setminus A$ (which is closed as $A$ is isolated) has planar measure $mK > 0$ and $p_i K \cap p_i \bar{N}_j = \emptyset$ for $1 \leq i \leq j$.*

*Proof.* Take a closed nowhere dense linear subset $K_i$ of the line segment $p_i D \cap L_i$ such that $m(D \cap p_i^{-1} K_i) > \pi - 2^{-i}$ and let $K = \bigcap_{i=1}^{\infty} p_i^{-1} K_i \cap D$ so that

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