EQUIVARIANT MAPS WITH NONZERO HOPF INVARIANT

Theodore Chang

In this paper we will derive strong necessary conditions for the existence of a torus equivariant map $f\colon S^{4d-1} \to S^{2d}$ with nonzero Hopf invariant. These conditions are expressed in terms of the topological weight system, as defined by Wu-yi Hsiang [6], of the torus actions on S^{4d-1} and S^{2d}. They imply, for example:

COROLLARY. *Suppose a torus T acts almost effectively (that is, with discrete ineffective kernel) on either S^{4d-1} or S^{2d}. If an equivariant map $f\colon S^{4d-1} \to S^{2d}$ exists with nonzero Hopf invariant, then either:*

(i) *$F(T,S^{4d-1}) \sim S^{4r-1}$ and $F(T,S^{2d}) \sim S^{2r}$, $0 \le r \le d$, and, except when $r = 0$ and $d = 1,2$,* rank $T \le d - r$,

or (ii) *$F(T,S^{4d-1}) = \emptyset$ and $F(T,S^{2d}) \sim S^{2r}$ or $F(T,S^{4d-1}) \sim S^{2r-1}$ and $F(T,S^{2d}) \sim S^0$, $0 < r \le d$, and* rank $T \le 3$ *if $r = 1,2$,* rank $T \le 2$ *if $r \ge 3$.*

Here $F(T,X)$ denotes the fixed point set of T acting on X and $X \sim Y$ means $H^*(X;Q) = H^*(Y;Q)$. All cohomology will be with rational coefficients.

1. When $f\colon S^{4d-1} \to S^{2d}$ has nonzero Hopf invariant, its mapping cone $M(f)$ will be a rational cohomology projective plane whose cohomology is generated by an element of degree $2d$; such a space will be called a $P^2(2d)$. When f is equivariant with respect to a torus T acting on S^{4d-1} and S^{2d}, $M(f)$ will inherit a T action. The cohomology structure of the possible fixed point sets of the T action on $M(f)$ are well known (see, for example, [2, p. 393]) and it follows:

PROPOSITION 1. *If T acts on S^{4d-1} and on S^{2d} and if $f\colon S^{4d-1} \to S^{2d}$ is equivariant with nonzero Hopf invariant, then either*

(i) *$F(T,S^{4d-1}) \sim S^{2r-1}$ and $F(T,S^{2d}) \sim S^{2r}$, $0 \le r \le d$,*

or (ii) *$F(T,S^{4d-1}) = \emptyset$ and $F(T,S^{2d}) \sim S^{2r}$ or $F(T,S^{4d-1}) \sim S^{2r-1}$ and $F(T,S^{2d}) \sim S^0$, $0 < r \le d$.*

When case (i) of Proposition 1 occurs we will say the T actions are of type (i); otherwise the T actions are of type (ii). Case (i) occurs when $F(T,M(f)) \sim P^2(2r)$ and case (ii) occurs when $F(T,M(f)) \sim \mathrm{pt} + S^{2r}$; these are the only two possibilities.

If T acts on a sphere S^n with $F(T,S^n) \sim S^r$ ($r = -1$ when $F(T,S^n) = \emptyset$) a (topological) weight is a corank 1 subtorus H so that $F(H,S^n) \sim S^q$ with $q > r$. Its multiplicity is $(q - r)/2$ which is always integer. The Borel formula states that the sum of the multiplicities of the weights is exactly $(n - r)/2$; in particular the collection of weights is finite. A local weight H can be identified with an element $\omega \in H^2(B_T)$ which is defined up to multiplication by a nonzero rational constant. We simply choose ω to be any generator of the kernel of the restriction

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