A STRONGER INVARIANT FOR HOMOLOGY THEORY

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1. INTRODUCTION

In this paper we show that in any homology theory which satisfies the Eilenberg-Steenrod axioms, the homology groups for compact polyhedral pairs satisfy an invariance much stronger than homotopy type invariance; it is called m-homotopy type invariance. The simplest example is the torus T^2 and the wedge of spheres $S^2 \vee S^1 \vee S^1$, which do not have the same homotopy type but do have the same m-homotopy type; therefore, they must have the same homology groups. This is a special case of Theorem 3.8, which begins to classify spaces by m-homotopy type.

The proof uses certain multiple valued functions which we have called m-functions. An m-function is finite valued, and each point of its graph is assigned a multiplicity which is an element of a fixed ring. The multiplicities satisfy an additivity condition which insures that locally as well as globally, the multiplicity is conserved with respect to variations in the domain variable.

M-functions were used in [5] to describe the intersections of two smooth simple closed curves in general position in the plane. As one curve undergoes a homotopy, intersections appear and disappear; one gets a weighted multiple valued function which associates with each homotopy parameter value a finite number of intersections, each labeled $+1$, -1 or zero depending on the orientation of the intersection.

This situation occurs again in studying fixed points, for one is looking for intersections of the graph of a function $f\colon X \to X$ with the diagonal of the space $X \times X$. Given a homotopy $f_t\colon X \to X$ one obtains an m-function $g\colon I \to X$ in which the points of $g(t)$ are the fixed points of f_t and their multiplicities are the degrees of the fixed points.

One can construct m-functions that are fundamentally different from any continuous function. For example, as part of the m-homotopy equivalence mentioned above we have an m-function from S^2 to T^2 that can be described as follows. If one puts a two-sphere in the (hollow) interior of a torus, there is a projection from T^2 onto S^2. The inverse of this projection is an m-function; a graph point has multiplicity $+1$, -1, or zero depending on how the radial ray from the sphere center intersects the torus at the point. Unlike any continuous function $S^2 \to T^2$ this m-function has degree one and is not null-homotopic. Another difference is that m-functions do not behave well under products; diagrams involving products may not commute, and there is no cup product in m-homology.

It is not difficult to do homology with m-functions [1]. The m-homology theory, together with some applications to fixed points of continuous functions, is also

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