THE HOCHSTER-ROBERTS THEOREM OF INVARIANT THEORY

George Kempf

Let G be an algebraic k-group acting morphically on an affine k-scheme, where k is a field. The quotient $G\backslash X$ is the affine k-scheme, whose regular functions are the regular functions on X, which are invariant under the action of G.

We will assume that G is linearly reductive; *i.e.,* any finite dimensional representation of G is completely reducible. The classical finiteness theorem of Hilbert asserts that, if X is a k-scheme of finite type, then $G\backslash X$ is also [5] and [10]. A remarkable modern discovery is the

THEOREM 0.1. (*Hochster-Roberts* [7]). *If* X *is a regular* k*-scheme of finite type, then* $G\backslash X$ *is a Cohen-Macaulay* k*-scheme of finite type.*

In this paper, I will give another proof of this theorem. My proof is a modification of their proof. In fact, it relies on the proof of

THEOREM 0.2. *Let* $A \subset B$ *be two integral domains which are finitely generated algebras over a field* k. *If* B *is regular and* B *is a pure* A*-module, then* A *must be a Cohen-Macaulay ring.*

This result was conjectured by Hochster and Roberts in their paper. Furthermore, both results were established by them in finite characteristics with weaker noetherian assumptions. As B is a pure A-module if A is a direct summand of B as an A-module, my proof shows that the only fact from invariant theory, used in the proof of Theorem 0.1, is that the G-invariant projection

$$\Gamma(X, \mathcal{O}_X) \to \Gamma(G\backslash X, \mathcal{O}_{G\backslash X})$$

is a $\Gamma(G\backslash X, \mathcal{O}_{G\backslash X})$-module homomorphism.

In Hochster-Roberts' proof, Theorem 0.1 was proven by a reduction to the graded case, which is apparently not possible for Theorem 0.2. Their reduction to the graded case uses the G-action and can provide valuable information about the normal behavior of $G\backslash X$ along any stratum in terms of the invariants of linear representations of reductive subgroups of G.

In this paper, I will first prove Theorem 0.2, next explain how it implies the Theorem 0.1 and lastly show how the theorem may be used for actually computing invariants.

1. SOME BACKGROUND

Let S be a regular noetherian scheme and $\mathcal{F}$ be a coherent $\mathcal{O}_S$-module. Let F denote the support of $\mathcal{F}$. Let Z be a closed subset of F. Recall that $\mathcal{F}$ is called

Received March 16, 1978. Revision received October 4, 1978.
Partly supported by N.S.F. Grant MPS 75-05578.

Michigan Math. J. 26 (1979).