NORMAL DIRECT SUMMANDS OF HYPOREDUCTIVE OPERATORS

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In [2], C. K. Fong showed that if S is a hyporeductive operator, N is normal, and if S is quasi-similar to N, then S is normal. In this paper we obtain an extension of Fong's result; in particular, we show that if there are any non-zero operators X and Y such that SX = XN and YS = NY, then S has a normal direct summand.

In what follows $\mathscr{H}$ will be a separable complex Hilbert space, N will be a fixed normal operator in $\mathscr{B}(\mathscr{H})$, and S will be a fixed *hyporeductive* operator in $\mathscr{B}(\mathscr{H})$; that is, S has the property that every hyperinvariant subspace reduces S.

If A and B are any two operators we will use the following notation:

$$\mathscr{L}(A,B) = \{Y: YA = BY\}$$
$$\mathscr{R}(A,B) = \{X: AX = XB\}.$$

(The letters $\mathscr{L}$ and $\mathscr{R}$ are chosen to reflect the position of Y or X with respect to A; in the defining equation Y appears on the left, X on the right of A.) For convenience we will refer to $\mathscr{L}(S,N)$ and $\mathscr{R}(S,N)$ as simply $\mathscr{L}$ and $\mathscr{R}$. $\mathscr{L}$ and $\mathscr{R}$ are not empty since the zero operator is in each. In addition, let $K_{\mathscr{L}}$ be the projection whose range is $\left[\bigcap \{\ker Y: Y \in \mathscr{L}\}\right]^{\perp}$ and let $R_{\mathscr{R}}$ be the projection whose range is $\bigvee \{\operatorname{ran} X: X \in \mathscr{R}\}$. Evidently, $\ker Y \supseteq \ker K_{\mathscr{L}}$ and $\operatorname{ran} X \subseteq \operatorname{ran} R_{\mathscr{R}}$ for each Y in $\mathscr{L}$ and X in $\mathscr{R}$.

THEOREM 1. *With the above notation, if* $K_{\mathscr{L}}$ *and* $R_{\mathscr{R}}$ *are both equal to the identity, then* S *is normal.*

Notice that Fong's result is a special case of Theorem 1, since if there exist quasiaffinities Y and X in $\mathscr{L}$ and $\mathscr{R}$ respectively, then $K_{\mathscr{L}} = R_{\mathscr{R}} = 1$ trivially. The proof below is based on the proof in [2].

Proof. First observe that if Y and X are in $\mathscr{L}$ and $\mathscr{R}$ respectively, and if C commutes with S and D commutes with N, then DY and YC are in $\mathscr{L}$ and XD and CX are in $\mathscr{R}$.

Suppose that $\mathscr{M}$ is a hyperinvariant subspace of the normal operator N. Denote

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