APPROXIMATION OF ANALYTIC FUNCTIONS SATISFYING A LIPSCHITZ CONDITION

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1. Let λ_α, $0 < \alpha < 1$, denote the class of functions f analytic in the open unit disk, continuous in the closed disk for which $t^{-\alpha}\omega(t) \to 0$ as $t \to 0$, where ω denotes the modulus of continuity of the boundary function of f.

Defining

$$\|f\|_\alpha = \|f\|_\infty + \sup t^{-\alpha}\omega(+) \tag{1.1}$$

yields a Banach algebra norm on λ_α. A theorem of Hardy and Littlewood [7, I, p.263] guarantees that $f \in \lambda_\alpha$ if and only if

$$|f'(z)| = o((1 - |z|)^{\alpha-1}) \qquad \text{as } |z| \to 1^-. \tag{1.2}$$

This theorem yields an equivalent Banach algebra norm on λ_α by setting

$$\|f\| = \|f\|_\infty + \sup \{(1 - |z|)^{1-\alpha}|f'(z)|: |z| < 1\}. \tag{1.3}$$

The norm given by (1.3) will be used exclusively in the sequel.

Every function $f \in \lambda_\alpha$ has a canonical factorization $f = FG$, where F is an outer function and G is an inner function. The purpose of this paper is to prove theorem A below, which states, in effect, that a function f in λ_α can be approximated by functions in λ_α with the same inner factor and with boundary zeros of arbitrarily high order.

THEOREM A. *Let* $f \in \lambda_\alpha$ *and let* E *be a closed set on the unit circle such that* $f(z) = 0$ *for all* $z \in E$. *Let* $M > 0$ *be given. Then for every* $\varepsilon > 0$ *there exists a function* $f_\varepsilon \in \lambda_\alpha$ *such that*

(i) *the inner factors of* f *and* f_ε *coincide,*

(ii) $\|f - f_\varepsilon\| < \varepsilon$, *and*

(iii) $|f_\varepsilon(z)| = O(\operatorname{dist}^M(z,E))$ *as* $\operatorname{dist}(z,E) \to 0$.

Theorem A can be used to give a characterization of the closed ideals in λ_α analogous to the Rudin-Beurling characterization of the closed ideals in the disc algebra [5]. The argument is similar to that of Korenblum [2] and is presented in [4].

The principal difficulty in the proof of Theorem A is isolated in the next theorem.

THEOREM B. *Let* $f \in \lambda_\alpha$ *be of the form* F^pG *where* F *is outer,* G *is inner,* $p > 1$ *and* $FG \in \lambda_\alpha$. *Let* Γ *be an open subset of the unit circle such that* f *vanishes at the endpoints of each component interval of* Γ. *Define*

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