NORM MAPS FOR FORMAL GROUPS IV

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1. INTRODUCTION

Let K be discretely valued complete field of characteristic zero with algebraically closed residue field k of characteristic $p > 0$. Let A be the ring of integers of K, and let F be a one-dimensional commutative formal group over A. Let K_∞/K be a $\mathbb{Z}_p$-extension (also called Γ-extension); *i.e.*, K_∞/K is Galois and $\mathrm{Gal}(K_\infty/K) \simeq \mathbb{Z}_p$, the p-adic integers. Let K_n be the invariant field of $p^n \mathrm{Gal}(K_\infty/K)$. There are natural norm maps $F - \mathrm{Norm}_{n/o}: F(K_n) \to F(K)$. Let v be the normalized exponential valuation on K; *i.e.*, $v(\pi) = 1$, where π is a uniformizing element of K. Let $F^s(K)$, $s \in \mathbb{R}$, $s \geq 1$, denote the filtration subgroup of $F(K)$ consisting of all elements x of A such that $v(x) \geq s$. Let h be the height of the formal group F and let e_K be the (absolute) ramification index of K; *i.e.*, $v(p) = e_K$. In [3] we proved:

THEOREM A. *There exist constants* c_1 *and* c_2 *such that for all* $n \in \mathbb{N}$, $F^{\beta_n}(K) \subset \mathrm{Im}(F - \mathrm{Norm}_{n/o}) \subset F^{\alpha_n}(K)$, *where*

$$\alpha_n = h^{-1}(h-1)ne_K - c_1, \ \beta_n = h^{-1}(h-1)ne_K + c_2.$$

The proof in [3] that there exists a constant c_1 such that the second inclusion holds is relatively easy, but the proof in [3] that there is a c_2 such that the first inclusion holds is very long and laborious. It is the purpose of the present note to give a much shorter and more conceptual proof of this part of the theorem by using some results on the logarithm of F. This proof is similar in spirit to the proof sketched in Section 12 of [3] for the main theorem of [2].

For more complete definitions of the notions mentioned above, see [2] and [3].

Here is some motivation for studying the images of norm maps for formal groups. Let $L - K - \mathbb{Q}_p$ be a tower of algebraic extensions of $\mathbb{Q}_p$ and let L/K be abelian galois. Then by local class field theory, $\mathrm{Gal}(L/K) \simeq K^*/N_{L/K}(L^*)$. The most interesting part (and the hardest to deal with) of this isomorphism is $\mathrm{Gal}(L/K)_1 = U^1(K)/N_{L/K}U^1(L)$, where $U^1(K)$ is the group of "Eins-Einheiten" of K; *i.e.*, $U^1(K) = 1 + \pi A$, and $\mathrm{Gal}(L/K)_1$ is the ramification subgroup of $\mathrm{Gal}(L/K)$ which corresponds to the wildly and totally ramified part of L/K of degree a power of p.

Now consider the multiplicative formal group $\hat{\mathbb{G}}_m(X,Y) = X + Y + XY$. Then $\mathbb{G}_m(K) = U^1(K)$, $\mathbb{G}_m(L) = U^1(L)$ and we see that the study of the norm maps $\hat{\mathbb{G}}_m - \mathrm{Norm}_{L/K}$ is what a not inconsiderable part of local class field theory is about.

Received June 4, 1976. Revisions received December 8, 1976 and July 18, 1977.

Michigan Math. J. 25 (1978).