A RESIDUALLY CENTRAL GROUP THAT IS NOT A Z-GROUP

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A group G is *residually central* if $x \notin [x,G]$ holds for every non-identity element x of G. Any Z-group (that is, one which has a central system in the sense of Kurosh [5, p. 218]) is residually central. John Durbin proved in [2] that a locally finite residually central group is a Z-group, and asked whether every residually central group G must be a Z-group (see also Robinson [8, p. 13]). He showed in [3] that this is true if G satisfies the minimum condition for normal subgroups, as did Ayoub in [1]. It is also true and not hard to see that residually central groups G which are either Abelian-by-nilpotent or Abelian-by-locally finite are Z-groups.

It occurred to us that a recent result of P. A. Linnell [6, Theorem A] could be used to give easily understood examples of residually central groups which are not Z-groups. Linnell has shown that if G is a torsion-free polycyclic group with an Abelian subgroup of finite index, then the group algebra KG over any field of nonzero characteristic has no zero divisors.

Suppose G is a nontrivial such group; suppose also that it is residually nilpotent and that G/G' is finite. Let p be a prime not dividing $|G/G'|$ and K the field with p-elements. We form the natural split extension $\Gamma = (KG) \rtimes G$; this, by a well known theorem of P. Hall [4], satisfies the maximum condition on normal subgroups. If $\mathfrak{g}$ denotes the augmentation ideal of KG, then $\mathfrak{g} = \mathfrak{g}^2$ by our choice of p; thus $\mathfrak{g}$ is the limit of the lower central series of Γ. It follows that Γ is not a Z-group.

On the other hand, Γ is residually central. For if x is nonzero in KG, then x cannot lie in $[x,\Gamma]$, since $[x,\Gamma] = x\mathfrak{g}$ and, by Linnell's theorem, an equation $x = x\delta$ cannot hold in KG unless $\delta = 1$. If x is in Γ but not in KG, then (KG)x does not lie in $(KG)[x,\Gamma]$ since G is residually nilpotent. Thus, Γ is residually central.

One example of a group G with all of the stated properties is the group

$$G = \langle x, y \colon x^{-1}y^2x = y^{-2},\ y^{-1}x^2y = x^{-2}\rangle$$

discussed by Passman [7, p. 96]. This group has the additional property of being supersoluble.

Thus, there is a finitely generated Abelian-by-supersoluble group Γ *which is residually central and is not a Z-group.*

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