BOUNDARY VALUE ESTIMATION OF THE RANGE OF AN ANALYTIC FUNCTION

D. L. Burkholder

Let S be a set of complex numbers and G a function analytic in $D = \{|z| < 1\}$. Denote the nontangential limit of G at $e^{i\theta}$, if it exists, by $G(e^{i\theta})$ and write $G(e^{i\theta}) \in S$ a.e. to mean that, for almost all θ, the limit does exist and belongs to S.

We seek conditions under which

(1) $$G(e^{i\theta}) \in S \text{ a.e.} \Rightarrow G(D) \subset S.$$

Neuwirth and Newman [12] showed that if S is the positive real axis, then (1) holds for all G in the Hardy class $H^{1/2}$. Hansen [8] gave a more general approach to such results and showed, among other things, that if S is the complement of the sector $\{re^{i\theta} : r \geq 0, 0 \leq \theta \leq \pi/p\}$, where $p > 1/2$, then (1) holds for all $G \in H^p$.

Now suppose that F is also analytic in D and $p > 0$. If $F(D) \cap G(D)$ is nonempty and $G(e^{i\theta}) \notin F(D)$ a.e., then

(2) $$G \in H^p \Rightarrow F \in H^p$$

[1, Theorem 4.4]. Hansen's method, somewhat strengthened, can be described as follows: If $\mathscr{F}$ is a family of functions analytic in D such that $\bigcup_{F \in \mathscr{F}} F(D)$ is a dense subset of $\mathbb{C} - S$ and $F \notin H^p$ for all $F \in \mathscr{F}$, then (1) holds for all $G \in H^p$. This assertion follows at once from the statement containing (2) and easily yields the above examples and related results.

It is classical that if S is the imaginary axis, then (1) holds for all $G \in H^1$. Tepper and Neuwirth [13], who also study (1), ask the question whether H^1 can be replaced by a larger class of functions.

We show here that in all of these examples H^p can be replaced by the larger class M^p defined below and give related results and extensions. For example, if S is compact and $\mathbb{C} - S$ is connected, then (1) holds for all $G \in M^{\log}$ (see Corollary 3). Also, if u is harmonic in the half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ and has a vanishing nontangential limit at almost every $x \in \mathbb{R}^n$, then, under a similar mild condition, u vanishes everywhere on $\mathbb{R}^{n+1}_+$ (see Theorem 3).

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