LACUNARITY AND LIPSCHITZ PROPERTIES IN TOTALLY DISCONNECTED ABELIAN GROUPS

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In this paper we are concerned with investigating two ways in which Lipschitz properties of a function interact with lacunarity properties of its Fourier series in the setting of a totally disconnected compact abelian group. The first part of this paper deals with the lacunarity condition which forces a local Lipschitz condition to become global. This has been investigated on the circle by Izumi, Izumi and Kahane [2] where it is found that the Hadamard lacunarity condition is precisely the right condition. In our setting a complete characterization is obtained only for a restricted class of totally disconnected groups. However, this class is large enough to contain the Cantor groups. The second part of this paper deals with the lacunarity condition which together with a Lipschitz condition forces absolute convergence of the Fourier series. This has been investigated on the n-dimensional torus by Benke [1] where the lacunarity condition is almost characterized. In the present setting analogous results are obtained.

Let G be a compact totally disconnected abelian group whose dual group Γ is countable. Then there is a sequence of open subgroups $G = G_0 \supset G_1 \supset \dots \supset \{0\}$ which forms a base of neighborhoods at the identity. Let $\{0\} = \Gamma_0 \subset \Gamma_1 \subset \dots \subset \Gamma$ be the sequence of annihilators. Since G is totally disconnected Γ is a torsion group and we may assume without loss of generality that the Γ_j are finite and that Γ_{n+1}/Γ_n is cyclic. The cyclicity condition is only needed in Theorem 4.

Definition. Let m_j be the cardinality of Γ_j. Define ρ on $G \times G$ as follows. If $x = y$ then $\rho(x,y) = 0$. If $x \neq y$ then

$$\rho(x,y) = m_{k(x,y)}^{-1}$$

where $k(x,y)$ satisfies $x - y \in G_{k(x,y)} \setminus G_{k(x,y)+1}$.

Proposition. *ρ is a translation invariant metric on* G.

Proof. If $x - y \in G_j \setminus G_{j+1}$ then $(x+z) - (y+z) \in G_j \setminus G_{j+1}$ and hence ρ is translation invariant.

Next, since $\{G_j\}_{j=0}^{\infty}$ is a neighborhood base $\rho(x,y) \neq 0$ for all $x \neq y$. Furthermore, since $x - y \in G_j \setminus G_{j+1}$ if and only if $y - x \in G_j \setminus G_{j+1}$ it follows that $\rho(x,y) = \rho(y,x)$.

To show the triangle inequality, by translation invariance, it suffices to show $\rho(x,0) \leq \rho(x,z) + \rho(z,0)$. Suppose $x \in G_j \setminus G_{j+1}$ then $\rho(x,0) = m_j^{-1}$. If $z \in G_{j+1}$ then $x - z \in G_j \setminus G_{j+1}$ and hence $\rho(x,z) = m_j^{-1}$ which gives the inequality. If $z \notin G_{j+1}$ then $\rho(z,0) \geq m_j^{-1}$ which also gives the inequality.

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