APPROXIMATING DIRECT INTEGRALS OF OPERATORS BY DIRECT SUMS

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1. INTRODUCTION

Although most operator theorists feel comfortable with direct sums of operators, many feel uneasy when confronted with direct integrals of operators. Direct integrals are natural analogues of multiplications by complex L^∞-functions on complex L^2-spaces; they are multiplications by operator-valued L^∞-functions on vector-valued L^2-spaces. One major problem is that complex L^∞-functions can be uniformly approximated by simple functions, but operator-valued L^∞-functions cannot even be approximated by functions with countable range. An operator-valued L^∞-function with countable range is really a direct sum (see Lemma D); therefore direct integrals cannot be "naturally" approximated by direct sums. It is the purpose of this note to show that, if one is willing to leave the measure-theoretic structure, direct integrals can be approximated by direct sums (Theorem A). As a consequence, many (but not all) questions about direct integrals can easily be answered without recourse to complicated measurability arguments. A sample question: Is the direct integral of quasidiagonal operators quasidiagonal? (A quasidiagonal operator [4] is one that is a limit of direct sums of finite matrices.) The affirmative answer to this question is a direct consequence of Theorem A (see Corollary B); a proof using more standard techniques (if one exists) would be a measure-theoretic nightmare.

Throughout, H denotes a separable Hilbert space and $(X,\mathscr{M},\mu)$ denotes a σ-finite measure space such that $L^2(\mu)$ is a separable Hilbert space.

Let $L^2(\mu;H)$ denote (the equivalence classes of) all functions $f\colon X \to H$ such that f is Borel measurable and $\int_X \|f(x)\|^2 d\mu(x)$ is finite. This space is also denoted by $\int_X^{\oplus} H\, d\mu(x)$ and is called the *direct integral* of H over X. It is proved in [1] that $L^2(\mu;H)$ is a Hilbert space with the inner product defined by

$$(f, g) = \int_X (f(x), g(x))\, d\mu(x).$$

Let B(H) denote the set of *operators* (bounded linear transformations) on H. Let $L^\infty(\mu; B(H))$ denote the set of all functions $\tau\colon X \to B(H)$ such that τ is weakly

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