SEMIGROUPS OF ANALYTIC FUNCTIONS AND COMPOSITION OPERATORS

Earl Berkson and Horacio Porta

Let U be an open set in the complex plane $\mathbb{C}$. A one-parameter semigroup $\{\phi_t\}$ of holomorphic mappings of U into itself is a homomorphism $t \mapsto \phi_t$ of the additive semigroup of nonnegative real numbers $\mathbb{R}^+$ into the semigroup (under composition) of all analytic mappings of U into U such that ϕ_0 is the identity map of U and $\phi_t(z)$ is continuous in (t, z) on $\mathbb{R}^+ \times U$. We also write $\phi(t, z)$ for $\phi_t(z)$, and denote $\dfrac{\partial \phi(t, z)}{\partial t}$ by $\phi_1(t, z)$. In this paper we study the collection $\mathscr{S}(U)$ of all such one-parameter semigroups on U for U the right half-plane or the open unit disc Δ, and then apply the results to a treatment of strongly continuous one-parameter semigroups of composition operators on $H^p(\Delta)$, $1 \leq p < \infty$.

In Section 1, we show for an arbitrary open set U that if $\{\phi_t\} \in \mathscr{S}(U)$, then there is a unique analytic function G on U (called the infinitesimal generator of $\{\phi_t\}$) such that $\phi_1(t, z) = G(\phi(t, z))$ on $\mathbb{R}^+ \times U$. In Section 2 we characterize and concretely describe the class of all infinitesimal generators for the case where U is the right half-plane. This involves proving the existence of a global solution to the initial value problem $\phi_1(t, z) = G(\phi(t, z))$, $\phi(0, z) = z$ for appropriate analytic functions G (see Theorems (2.6) and (2.13) below). In Section 3, after rephrasing these results so as to characterize the generators for the case where U is Δ, we study the strongly continuous one-parameter semigroups of composition operators on $H^p(\Delta)$, $1 \leq p < \infty$, and characterize their infinitesimal generators in Theorem (3.7). For $1 \leq p < \infty$, every $\{\phi_t\} \in \mathscr{S}(\Delta)$ gives rise to a strongly continuous semigroup $\{T_t\}$ of composition operators on $H^p(\Delta)$. The point spectrum of the infinitesimal generator of $\{T_t\}$, in certain cases, is taken up in Section 4, where an interplay with logarithmic potentials develops.

Throughout what follows, we denote composition of mappings by $\circ$ and differentiation with respect to z by $'$.

The authors are indebted to Professor Robert Kaufman for decisive contributions including the framework of Sections 1 and 2.

1. THE INFINITESIMAL GENERATOR OF A SEMI-GROUP OF HOLOMORPHIC MAPPINGS

(1.1) THEOREM. *Let* U *be an open set in* $\mathbb{C}$, *and let* $\{\phi_t\}$, $t \in \mathbb{R}^+$, *be a one-parameter semigroup of holomorphic mappings of* U *into* U. *Then there is*

Received November 1, 1976.
The first author was supported by a National Science Foundation grant.

Michigan Math. J. 25 (1978).