OMITTED VALUES OF SINGULAR INNER FUNCTIONS

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In this paper we investigate certain properties of measures on the unit circle T associated with singular inner functions which omit values in the unit disc U. Our results are used to resolve some open questions concerning inner functions; in particular, we disprove a conjecture of Herrero concerning the structure of the inner functions under the uniform topology of H^∞, the space of bounded analytic functions on U. We assume that the reader is familiar with the basic theory of H^∞, the notion of logarithmic capacity for plane sets, and the elementary properties of universal covering surfaces for plane regions. Appropriate references would be Duren [5], Tsuji [10], and Ahlfors [1], respectively.

We briefly describe our main results below. The preliminary material is discussed in more detail and notations are established in Section 2.

1. MAIN RESULTS

If A is a (relatively) closed subset of U with (logarithmic) capacity zero, then the universal covering surface of $U \setminus A$ is conformally equivalent to U. If ϕ_A is an uniformizer of $U \setminus A$ (see 2.3), then ϕ_A is an inner function whose range is precisely $U \setminus A$. For our main result we assume $0 \in A$, so that ϕ_A is a singular inner function.

THEOREM I. *Let* A *be a closed subset of* U *of (logarithmic) capacity zero,* $0 \in A$, *and let* μ *be the singular measure on* T *associated with the conformal mapping* ϕ_A *of* U *onto* $U \setminus A$.

(a) *If* 0 *is an isolated point of* A, *then* μ *is discrete; i.e., it consists entirely of point masses.*

(b) *If* 0 *is a limit point of* A, *then* μ *is continuous; i.e., it has no point masses.*

The proofs of parts (a) and (b) require entirely different techniques and are given in Sections 3 and 4, respectively. Part (b) is actually a corollary to a stronger result, Theorem 4.2. The main ingredient is a mapping theorem which may be of some independent interest:

THEOREM II. *Let* F *be an analytic function from* U *into the left half-plane with the property that* $\liminf_{r \to 1} (1-r)|F(r)| > 0$. *Then, for each* $M > 0$, *the disc* $\{w \in \mathbb{C} : |w - F(r)| < M\}$ *lies in the range of* F *for all* r *sufficiently close to* 1, $0 < r < 1$.

This study was originally motivated by certain conjectures of Herrero [8] concerning inner functions under the uniform norm of H^∞. In Section 5 we use

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