INTEGRAL REPRESENTATIONS AND DIAGRAMS

E. L. Green and I. Reiner

INTRODUCTION

This article is the outcome of an attempt to study integral representations by diagrammatic techniques. A *diagram* D is a finite directed graph. Given a field k, a k-*representation* of D assigns to each vertex α of D a finite dimensional k-space V_α, and to each arrow $\alpha \to \beta$ a k-linear transformation $V_\alpha \to V_\beta$. There are obvious definitions of morphisms of representations, isomorphisms, direct sum, and decomposability. It is clear that the Krull-Schmidt Theorem is valid for representations of diagrams, namely, every representation is expressible as a finite direct sum of indecomposables, unique up to isomorphism and order of occurrence. If there are only a finite number of non-isomorphic indecomposables, we call D of finite representation type.

These concepts were introduced in a fundamental article by Gabriel [9], who proved that a connected diagram D is of finite type if and only if its underlying graph is one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8. Some time thereafter, a less computational proof of this amazing result was given by Bernstein-Gelfand-Ponomarev [1], using the machinery of Coxeter functors from Lie algebras. Their approach was generalized by Dlab-Ringel [4], [5], who considered representations of a modulated graph $\mathcal{M}$. By definition, $\mathcal{M}$ consists of a finite directed graph with a skewfield k_α placed at each vertex α, and a (k_β, k_α)-bimodule ${}_\beta M_\alpha$ attached to each arrow $\alpha \to \beta$. A representation of $\mathcal{M}$ assigns to each vertex α a left k_α-space V_α, and to each arrow from α to β a k_β-homomorphism ${}_\beta M_\alpha \otimes_{k_\alpha} V_\alpha \to V_\beta$. After imposing a few reasonable hypotheses, Dlab-Ringel determined all modulated graphs of finite type.

A somewhat different approach was followed by Russian mathematicians such as Drozd, Kleiner, Nazarova, and Roiter (see the fundamental Leningrad Proceedings of 1972 [16]). They studied finite partially ordered sets (posets); a representation of a poset S is given by choosing a vector space V over some field k, and assigning to each $\alpha \in S$ a subspace V_α of V, so that $V_\alpha \subseteq V_\beta$ whenever $\alpha \leq \beta$ in S. The work of Nazarova-Roiter and Kleiner settled the question as to which posets have finite type.

It has become increasingly clear from the above-mentioned articles, as well as from related work of Donovan-Freislich [6], Gordon-Green [11], Green [12], [13], and Ringel [18], [19], that these diagrammatic methods provide a new and powerful tool for investigating representations of rings and algebras. Such

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