AN ARC IN A PL n-MANIFOLD WITH NO NEIGHBORHOOD THAT EMBEDS IN S^n, $n \geq 4$

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1. INTRODUCTION

To study the geometric embedding properties of a compactum X in a PL n-manifold, one sometimes finds it useful to change the scenery by reembedding a neighborhood of X in the n-sphere S^n. For example, if $n = 3$ and if X is a topological cell, this can always be done. (See [9] for a proof; the case of an arc was also done in [1].) The ability to make this transition in many instances greatly increases the usefulness of the "cellularity criterion" in 3-manifolds, for example. (See [9].)

It seems reasonable to expect that such familiar neighborhoods should be found whenever X is a "nice" compactum embedded in a PL n-manifold. Unfortunately, this is not so. It fails at the first opportunity: There is, for each $n \geq 4$, an arc A embedded in a certain PL n-manifold M^n such that no neighborhood of A in M^n embeds topologically in S^n. Our intuition is further violated by the fact that each proper subarc of A has a neighborhood that embeds in S^n.

Our basic four-dimensional construction is motivated by A. Casson's example of a cell-like continuum in a 4-manifold. Robert J. Daverman showed us this earlier construction, which seems by now to be widely known. It may be a counterexample to the four-dimensional cellularity criterion, at least in the smooth setting. No proof of this yet exists, however, and our methods definitely fail on it. We also wish to acknowledge that Robert D. Edwards suggested using decomposition space techniques (such as those of [5] and [12]) to shrink the Casson example down to an arc. Our overall approach owes much to these sources. We also thank Bob Sternfeld and Mike Starbird for many helpful discussions. In particular, Starbird showed us how to free our examples for $n > 4$ (Section 4) from relying on an as-yet-unpublished higher-dimensional version of the Edwards-Miller-Pixley-Eaton theorem.

Here is a brief summary of our notation. I denotes the unit interval [0,1]. B^n is $[-1,1]^n$; S^n is the n-sphere; and E^n is Euclidean n-space. A *loop in* X is a (continuous) mapping $S^1 \to X$. Integer coefficients are understood for (co) homology and $\tilde{H}$ denotes reduced (co) homology. Isomorphism of groups A, B is symbolized by $A \cong B$, and homeomorphism of spaces X, Y by $X \approx Y$. We work in the PL context throughout. A *cube-with-handles* is any PL homeomorph of the regular neighborhood in S^3 of a finite connected graph. Its *genus* is: one minus its Euler characteristic.

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