APPROXIMATING DISKS IN 4-SPACE

Gerard A. Venema

1. INTRODUCTION

In this paper we show that certain topological embeddings of the $(n - 2)$-disk into a PL n-manifold can be approximated by locally flat PL embeddings. The problem of approximating topological embeddings has been studied extensively and approximation theorems are already known in all codimensions other than two: Miller [8] proved that all topological embeddings of manifolds can be PL approximated in codimensions greater than or equal to 3, and Ancel and Cannon [1] have recently used a technique of Stankŏ to prove a locally flat approximation theorem for manifolds in codimension 1. Our main theorem applies to 2-disks embedded in a 4-manifold.

THEOREM 1. *If* $D\colon I^2 \to M^4$ *is a topological embedding of a disk into a* PL 4-*manifold, then* D *can be* ε-*approximated by a locally flat* PL *embedding* $E\colon I^2 \to M^4$ *for every* $\varepsilon > 0$. *Furthermore, if* $D|\partial I^2$ *is* PL *and* $D(\partial I^2) \subset \operatorname{Int} M^4$, *then* E *can be chosen so that* $E|\partial I^2 = D|\partial I^2$.

If $D(\partial I^2) \subset \partial M^4$, we cannot have $E|\partial I^2 = D|\partial I^2$. For example, if D is the cone from the center of the 4-ball B^4 to a trefoil knot on ∂B^4, then ∂D does not bound a locally flat PL disk in B^4 [6]. However, it is still unknown whether it is possible to have $D(\partial I^2) \subset \partial M^4$ in this case.

It is natural to ask whether Theorem 1 is true when I^2 is replaced by some other 2-manifold, since in codimension 3 an approximation theorem for manifolds follows from one for disks. In general it is not; an example is given in [5] of a topological embedding of the 2-torus $S^1 \times S^1$ into the 4-sphere S^4 which cannot be approximated arbitrarily closely by PL embeddings (not even by PL embeddings with non-locally flat points). The answer is unknown for embeddings of S^2 in S^4. The following theorem, which can be proved in exactly the same fashion as Theorem 1, gives a positive answer for certain 2-complexes.

THEOREM 2. *If* K *is a finite* 1-*complex and* $h\colon K \times I \to M^4$ *is a topological embedding, then* h *can be approximated by* PL *embeddings.*

Any topological disk in a topological 4-manifold has a neighborhood which can be immersed in $\mathbb{R}^4$ [7]. This neighborhood inherits a PL structure from $\mathbb{R}^4$ and so Theorem 1 can be used to find an approximation which is locally flat and PL with respect to the inherited structure. Since local flatness is a topological property, the following theorem is a consequence of Theorem 1.

THEOREM 3. *Any topological embedding of the* 2-*disk into a topological* 4-*manifold can be* ε-*approximated by locally flat embeddings for every* $\varepsilon > 0$.

Received September 17, 1976. Revision received May 31, 1977.

Michigan Math. J. 25(1978).