ON BANACH SPACES WITH UNIQUE ISOMETRIC PREDUALS

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A Banach space Y is called an isometric predual, or simply a predual, of a Banach space X if the dual Y^* of Y is isometrically isomorphic to X. A Banach space X is said to have a unique isometric predual if X has a predual and all preduals are mutually isometrically isomorphic. A. Grothendieck [5] first noticed the uniqueness of isometric preduals of L^∞-spaces and later S. Sakai generalized this to von Neumann algebras, see [7]. In this note a further generalization of Sakai's result will be proved. Evidently, infinite dimensional von Neumann algebras are the only previously known class of non-reflexive Banach spaces with unique isometric preduals; our generalization supplies a slightly broader class of such Banach spaces.

First we state one known lemma which is essentially due to Dixmier [2].

LEMMA 1. *Let* X *be a Banach space and let* X^* *be its dual.*

(a) *Suppose* Z *is a closed subspace of* X^* *satisfying the following two conditions;*

(i) Z *is total over* X *and minimal with respect to the property of being norm closed and total over* X,

(ii) Z *norms* X, *that is* $\|x\| = \sup\{\phi(x)\colon \phi \in Z,\ \|\phi\| \leq 1\}$ *for all* $x \in X$.

Then X *is the dual of* Z *in the canonical way. Conversely any predual* Y *of* X *is isometrically isomorphic to a closed subspace* Z *of* X^* *having* (i) *and* (ii) *above.*

(b) *Statements* (i) *and* (ii) *above are together equivalent to*

(iii) *the closed unit ball of* X *is compact with respect to the weak topology on* X *induced by* Z.

A proof of (a) can be given using the Hahn-Banach theorem, and (b) is a direct consequence of Alaoglu's theorem and the bipolar theorem.

By definition, a von Neumann algebra X acting on a Hilbert space H is an algebra consisting of bounded operators on H, invariant under the adjoint operation, containing the identity operator 1_H on H and closed in the weak operator topology on the space of all bounded operators on H. The σ-weak operator topology on X is defined by the family of semi-norms $p(x) = \sum_{n=1}^{\infty} |\langle x\xi_n, \eta_n \rangle|$ $(x \in X)$, where $\{\xi_n\}$ and $\{\eta_n\}$ are sequences in H with $\sum_{n=1}^{\infty} \|\xi_n\|^2, \sum_{n=1}^{\infty} \|\eta_n\|^2 < +\infty$. A linear functional ϕ on X is called *normal* if ϕ is continuous in the σ-weak operator topology. The space of all normal linear functionals on X is denoted by X_*. It is well known that X_* is a closed subspace of the dual X^* of X, X is the dual of X_* in the canonical way and the σ-weak operator topology on X is equal to the weak-* topology on X as the dual of X_*, see [4].

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