VECTOR MEASURES AND SCALAR OPERATORS IN LOCALLY CONVEX SPACES

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P. G. Spain [15] has determined two different necessary and sufficient conditions for a bounded linear operator on a Banach space with an operational calculus defined for continuous functions on its spectrum to be a scalar-type spectral operator. B. Walsh [17] and M. Tidten [16] have discussed spectral measures whose range is an equicontinuous family of projections on a locally convex space, in order to study operators on nuclear spaces. In this paper, we show how Spain's problem is related to the general theory of vector measures and solve it in the setting of Walsh and Tidten. Spain's two conditions turn out to be valid for locally convex spaces, but under different completeness hypotheses (Theorems 5 and 6 below). Some of our results are new even for Banach spaces.

In order to avoid restricting our attention to barreled spaces, we assume, as in [17] and [16], that certain sets of linear functions are equicontinuous rather than merely bounded. An alternate approach is to view this problem from the standpoint of bornology theory [2] and consider morphisms of b-algebras rather than certain continuous linear mappings of locally convex spaces, but we shall not do this.

Throughout this article, K denotes a compact Hausdorff space, C(K) [B(K)] the Banach space of continuous [bounded Borel-measurable] complex functions on K, $U\,[U_1]$ the unit ball in C(K) [B(K)], $\Sigma(K)$ the σ-algebra of Borel sets of K and E a locally convex Hausdorff space. In general, we follow the terminology and notation of Schaefer [12] for topological vector spaces, but we use L(E) rather than $\mathscr{L}(E)$ to denote the space of continuous linear operators on E.

A *(vector) measure* is a countably additive set function $\mu: \Sigma(K) \to E$. By the Orlicz-Pettis theorem, μ is countably additive for the weak topology of E if and only if it is for the initial topology. A measure is regular for the weak topology if and only if it is for the initial one [9, Theorem 1.6]. The range of a measure is a bounded set [9, p. 158]. If $\mu: \Sigma(K) \to E$ is a measure, then $\int f\,d\mu \in E$ can be defined in the obvious way for simple functions in B(K) and is continuous in f. If E is sequentially complete, we define $\Pi f = \int f\,d\mu$ for $f \in B(K)$ by continuity, as an element of E. Then $\Pi: B(K) \to E$ is a continuous linear map and if Φ denotes its restriction to C(K) and μ is a regular measure, then Π is the restriction of Φ'' from $C(K)''$ to its subspace B(K) [13, Theorem 1]. Since C(K) is dense in $C(K)''_\sigma$ and $\Phi'': C(K)''_\sigma \to E''_\sigma$ is continuous (E''_σ denotes E'' with the topology $\sigma(E'', E')$), for each $f \in B(K)$ there is a net (f_α) in C(K) such that $\int f_\alpha\,d\mu \to \int f\,d\mu$ weakly in E. In fact, the following (which may be new even for Banach spaces) shows we can obtain convergence in the initial topology of E. If we know in advance that the integrals in question are elements of E, we can drop the completeness assumption.

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