A NOTE ON THE $\mathbb{X}$-OPERATOR OF AN $[FIA]^-$ GROUP

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An $[FIA]^-$ group is a locally compact group G for which the set of inner automorphisms I(G) is relatively compact in the (topologized) group Aut(G) of all topological automorphisms of G. For such a group, Mosak [4] has constructed a mapping $a \to a^{\mathbb{X}}$ from the C^*-algebra $C^*(G)$ onto its center Z. This mapping is a center-valued trace, and in this note we show that it is related to an averaging process in $C^*(G)$ similar to Dixmier's process for obtaining the trace in a finite von Neumann algebra [1; Chapitre III, Sections 5, 8].

We begin with some definitions and notation. G will be understood to be an $[FIA]^-$ group, A will denote its C^*-algebra $C^*(G)$, and $\tilde{A}$ will denote the C^*-algebra obtained by adjoining an identity 1 to A (except in the special situation where A has an identity, in which case $\tilde{A}$ should be taken as equal to A). We will denote closure in the norm topology of A by norm-$c\ell$, and convex hull by co. If $\beta \in \mathrm{Aut}(G)$ and f is a complex valued function on G, then $f^\beta(x) = f(\beta^{-1} x)$ $(x \in G)$. If f is continuous, then

$$f^{\mathbb{X}}(x) = \int_{I(G)^-} f^\beta(x)\, d\beta$$

where $d\beta$ is the normalized Haar measure on the compact group $I(G)^-$. For further details, including the definition of a^β and $a^{\mathbb{X}}$ when $a \in A$, we refer the reader to [4].

THEOREM. *For each* $a \in A$, $a^{\mathbb{X}} \in$ norm-$c\ell$ co $\{u^* a u\colon u$ unitary in $\tilde{A}\}$.

Proof. Since the $\mathbb{X}$-operator is linear and norm-decreasing [4; 1.5], a simple approximation argument shows that it suffices to consider the case where a is a continuous complex-valued function on G with compact support. By [4; 1.4],

$$(1) \qquad a^{\mathbb{X}} \in \text{norm-}c\ell\ \text{co}\,\{a^\beta\colon \beta \in I(G)^-\}\,.$$

Furthermore, since $I(G)^-$ is a subgroup of Aut(G), it follows from [4; 1.0] that if $\alpha \in I(G)^-$ then

$$(2) \qquad a^\alpha \in \text{norm-}c\ell\,\{a^\beta\colon \beta \in I(G)\}\,.$$

In view of (1) and (2), it suffices to fix $\beta \in I(G)$ and show that

$$(3) \qquad a^\beta \in \text{norm-}c\ell\ \text{co}\,\{u^* a u\colon u \text{ unitary in } \tilde{A}\}\,.$$

Let π be the universal representation of A (extended to $\tilde{A}$ by defining $\pi(1) = 1_\pi$), and let π also denote the associated unitary representation of G. Let $x \in G$ be such that $\beta(g) = x^{-1} g x$ $(g \in G)$. From the proof of [4; 1.2] and the discussion of $\Delta(\beta)$ which follows it, we obtain $\pi(a^\beta) = \pi(x)^* \pi(a) \pi(x)$. Since (by the double commutant theorem) $\pi(x)$ lies in the von Neumann algebra generated by $\pi(A)$,

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