ZEROS OF FUNCTIONS IN WEIGHTED BERGMAN SPACES

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1. INTRODUCTION

Throughout this paper, Δ denotes the open disc in the complex plane, and μ is a finite, positive Borel measure on the half-open interval $[0, 1)$, giving nonzero mass to each interval $[r, 1)$ for $0 \leq r < 1$. In other words, μ places no mass at the point 1, but has 1 in its closed support. We denote the rotation-invariant extension of μ to Δ by μ_Δ: this is the unique finite (positive) Borel measure on Δ such that

$$\int f d\mu_\Delta = \int \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{it})dt \right\} d\mu(r)$$

for each f continuous on the closed unit disc.

For $0 < p < \infty$, the *Bergman space* A^p_μ is the collection of functions f holomorphic in Δ such that $\|f\|_p^p = \int |f|^p d\mu_\Delta < \infty$. This paper deals with the zero sets of functions in A^p_μ. For f holomorphic in Ω and not $\equiv 0$, the *zero set of* f *in* Ω, denoted by Z(f) (or by $Z_\Omega(f)$ when we wish to emphasize the domain of f) is the sequence of zeros of f in Ω, ordered by increasing moduli, and repeated according to multiplicity. If $f \in A^p_\mu$ then we call Z(f) an A^p_μ*-zero set.*

We are going to show that *for* $0 < p < \infty$ *there is an* A^p_μ *zero set not contained in any* A^q_μ *zero set for* $q > p$; and that A^p_μ *has a disjoint pair of zero sets whose union is not a zero set.* Both these results have recently been obtained by Charles Horowitz [7] for the case $d\mu(r) = (1 - r)^\alpha dr$ $(\alpha > -1)$, and the first one has been proved for the Hardy spaces of balls and polydiscs in $\mathbb{C}^n$ by J. Miles [10] and W. Rudin [11]. While Horowitz and Miles employ infinite products to produce the required zero sets explicitly, Rudin constructs gap series with prescribed growth properties and leaves the zeros to fend for themselves. Rudin's idea plays the fundamental role in this paper: we will see that it provides a simple and flexible means for "constructing" the desired zero sets in Bergman spaces.

In particular, our results have relevance to operator theory. It is known that on Hilbert space every injective subnormal unilateral weighted shift not similar to the standard unweighted one is unitarily equivalent to the operator of multiplication by z on some A^2_μ (the standard unweighted shift is unitarily equivalent to multiplication by z on the Hardy space H^2) [2; Theorem 8], [13; Proposition 25, page 84]. It follows quickly from this and the existence of two disjoint A^2_μ zero sets whose union lies in no zero set that every such weighted shift has a pair of nontrivial invariant subspaces with trivial intersection.

It is well known that all these results fail in the Hardy spaces H^p, which can be regarded as the limiting case where μ has unit mass at $\{1\}$. Since a sequence

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