EVERY CRUMPLED n-CUBE IS A CLOSED n-CELL-COMPLEMENT

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Often it is a convenient simplification, in studying the wildness of (n - 1)-spheres topologically embedded in the n-sphere S^n, to suppose that the wildness is confined to one complementary domain, in that the closure of the other complementary domain is an n-cell. The principal result here, Theorem 6.1, shows that for $n \geq 5$ this simplification has validity in a stronger setting: for each crumpled n-cube C in S^n and $\varepsilon > 0$ there exists an ε-homeomorphism h of C into S^n such that the closure of S^n - h(C) is an n-cell. For n = 3 the same result has been established by Hosay [16] and Lininger [17].

A *crumpled* n-*cube* C is a space homeomorphic to the union of an (n - 1)-sphere in S^n and one of its complementary domains; the subset of C consisting of those points at which C is an n-manifold (without boundary) is called the *interior of* C, written Int C, and the subset C - Int C, which corresponds to the given (n - 1)-sphere, is called the *boundary of* C, written Bd C. A crumpled n-cube C is a *closed* n-*cell-complement* if there exists an embedding h of C in S^n such that S^n - h(Int C) (equivalently, the closure of S^n - h(C)) is an n-cell. Translated into this terminology, the principal result implies that for $n \geq 5$ each crumpled n-cube is a closed n-cell-complement (Corollary 6.4).

Besides validating this simplification, the paper supports the opposite process permitting the construction of complexities. To describe the construction, we look first at a standard situation: any (n - 1)-sphere Σ in S^n bounds two crumpled n-cubes C_0 and C_1, and fastening C_0 and C_1 (abstractly conceived) back together along their boundaries in an appropriate way reproduces S^n, with both Bd C_0 and Bd C_1 identified as Σ. Generally, such an attaching is called a sewing; specifically, a *sewing* of two crumpled n-cubes C_0 and C_1 is a homeomorphism between their boundaries, and associated with a sewing h is the *sewing space*, denoted as $C_0 \cup_h C_1$, which is the quotient space obtained from the disjoint union of C_0 and C_1 under identification of each point x in Bd C_0 with its image h(x) in Bd C_1. To construct (n - 1)-spheres in S^n with wildness in both complementary domains, one first can select crumpled n-cubes C_0 and C_1, together with a sewing h, and then can hope to prove that $C_0 \cup_h C_1$ is homeomorphic to S^n. The ultimate problem, determining whether a given sewing space $C_0 \cup_h C_1$ is topologically equivalent to S^n, can be an intricate and complex puzzle, about which [14] supplies much information. In any event, the results here (see Corollary 6.7) imply that the sewing space $C_0 \cup_h C_1$ is homeomorphic to a decomposition space associated with a decomposition of S^n into points and flat arcs, thereby reducing the sewing problem to a decomposition problem.

Instrumental for the approach used here is Ancel and Cannon's recent solution [1] of the Locally Flat Approximation Theorem, which states that each embedding of an (n - 1)-manifold in an n-manifold ($n \geq 5$) can be approximated arbitrarily closely by locally flat embeddings. Earlier Bryant, Edwards and Seebeck [5] had devised a

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