RATIONAL APPROXIMATION AND SWISS CHEESES

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1. INTRODUCTION

For the purpose of this note, an open set A of the complex plane will be called *regular* if its boundary ∂A is a finite union of piecewise C^2 simple closed curves. A compact set X of the plane will be called *areally disconnected* if for each $\alpha > 0$ there exist a finite number of pairwise disjoint regular open sets $A_1, \cdots, A_n$ (depending on α) for which $X \subset \bigcup_{k=1}^{n} A_k^-$ and, for $k = 1, \cdots, n$, $m_1(\partial A_k \cap X) = 0$ and $m_2(A_k \cap X) < \alpha$. (Here A_k^- denotes the closure of A_k, while m_1 and m_2 refer to Lebesgue arc length and planar measures.)

For any compact set X, let C(X) and R(X) denote, respectively, the algebra of continuous functions on X and the subalgebra of functions which are uniformly approximable on X by rational functions with poles off X. An obvious necessary condition in order that C(X) = R(X) is that the interior of X be empty. Various necessary and sufficient conditions for the validity of this equality are, in fact, known (*e.g.*, as consequences of Bishop's peak point criterion, Melnikov's peak point criterion, or Vitushkin's theorem; see Gamelin [4] or Zalcman [16]), but are not always easy to apply. A sufficient condition for C(X) = R(X) is contained in the following.

THEOREM 1. *If* X *is a compact set of the plane which is areally disconnected, then* C(X) = R(X).

COROLLARY 1 (Hartogs-Rosenthal [6]). *If* X *is a compact set of the plane and if* $m_2(X) = 0$, *then* C(X) = R(X).

The hypothesis $m_2(X) = 0$ implies that, for almost all real t, the line $\Re(z) = t$ intersects X in a set of zero linear measure. Hence, for each $\alpha > 0$, there exist a finite set of real numbers $t_0 < \cdots < t_n$ and a pair of real numbers $a < b$ satisfying $t_k - t_{k-1} < \alpha/(b - a)$ for $k = 1, \cdots, n$, with the properties that X is contained in the rectangle $(t_0, t_n) \times (a, b)$ and each segment $\{z: \Re(z) = t_k \text{ and } a \leq \Im(z) \leq b\}$, for $k = 0, \cdots, n$, intersects X in a set of zero linear measure. If the open rectangles $(t_{k-1}, t_k) \times (a, b)$, for $k = 1, \cdots, n$, are identified with the sets A_k considered at the beginning of this section, it is seen that X is areally disconnected.

It is clear from the above proof that Corollary 1 can be strengthened to the following.

COROLLARY 2. *If* X *is a compact set of the plane and if there exists a set of real numbers* $\{t\}$ *dense on the real line for which each of the vertical lines* $\Re(z) = t$ *intersects* X *in a set of zero linear measure, then* C(X) = R(X).

Obviously, the special role of the vertical direction in Corollary 2 is one of convenience, and any fixed direction would serve as well. It is noteworthy that, although the usual modern proofs of the Hartogs-Rosenthal theorem (see, *e.g.*, [4, p. 47] or [16, p. 110]), or even the original proof in [6], do not seem to yield Corollary

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