REGULAR NEIGHBORHOODS IN TOPOLOGICAL MANIFOLDS

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Regular neighborhoods have proved to be a very useful tool in the theory of PL manifolds. In this paper we want to make a very easy construction of regular neighborhoods in the topological category. F. E. A. Johnson [6] has constructed regular neighborhoods in topological manifolds, but only in the case of nonintersection with the boundary. R. D. Edwards [4] has announced a very general construction of regular neighborhoods; see also [3]. The present construction has the advantage of allowing a "relative" version, (Theorem 13), in the sense that if L is a complex, K is a subcomplex, and L is locally tamely embedded in a topological manifold V, then one may find a regular neighborhood of K in V, intersecting L in a regular neighborhood of K in L, in the usual PL sense. This is used in [10] to prove embedding theorems for topological manifolds. In [11] we have a proof that the opposite procedure is possible; namely, finding a spine of a topological manifold.

We should emphasize that the regular neighborhoods we obtain are mapping cylinder neighborhoods; *i.e.*, if $K \subset N$, where N is a regular neighborhood of K, then there is a map $\pi: \partial N \to K$ such that N is homeomorphic to the mapping cylinder of π (Theorem 15).

Let K be a compact topological space with a given simple homotopy structure; *i.e.*, of the homotopy type of a finite CW-complex, with the homotopy equivalence specified up to torsion.

Definition 1. A *regular neighborhood* N_2 of K in V is a locally flat, compact submanifold of V, of codimension 0, which is a topological neighborhood of K such that the inclusion $K \subset N$ is a simple homotopy equivalence, and K is a strong deformation retract of N. We also require that $\partial N \subset N - K$ induces an isomorphism on the fundamental group for every component.

Definition 2. A regular neighborhood N of $K \subset V$ is said to *meet the boundary regularly* if $N \cap \partial V$ is a regular neighborhood of L in ∂V and $\eta(N) = \overline{\partial N - N \cap \partial V}$ meets ∂V transversally.

Remark 3. If a regular neighborhood meets the boundary regularly, it then follows from van Kampen's theorem that $\eta(N) \to N - K$ induces an isomorphism on the fundamental group.

Definition 4. $K \subset V$ is said to have *arbitrarily small* regular neighborhoods if for every neighborhood U of K there is a regular neighborhood N of K in V such that $N \subset U$.

Definition 5. Two regular neighborhoods of $K \subset V$, N and $\tilde{N}$, are said to be *equivalent* if N is homeomorphic to $\tilde{N}$ by a homeomorphism which is the identity on a neighborhood of K. If N and $\tilde{N}$ meet the boundary regularly, the homeomorphism is required to restrict to a homeomorphism of $N \cap \partial V$ to $\tilde{N} \cap \partial V$.

We now want to change a regular neighborhood into one that meets the boundary regularly.

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