ON RIESZ TRANSFORMS OF BOUNDED FUNCTIONS OF COMPACT SUPPORT

Nguyen Xuan Uy

1. Let K be a compact set of $\mathbb{R}^n$ such that $m(K) > 0$, where m is n-dimensional Lebesgue measure. Let $L^\infty(K)$ denote the set of all functions in $L^\infty(\mathbb{R}^n)$ which vanish almost everywhere on $\mathbb{R}^n \setminus K$. We will be concerned with the set $H^\infty(K)$ of those functions in $L^\infty(K)$ which have bounded Riesz transforms. More precisely, a function $h \in L^\infty(K)$ is in $H^\infty(K)$ if and only if all Riesz transforms

$$R_j h(x) = \text{P.V. } c_n \int \frac{(x_j - t_j)}{|x - t|^{n+1}} h(t)\, dt, \qquad j = 1, 2, \cdots, n,$$

where c_n is normalizing constant depending only on n, belong to $L^\infty(\mathbb{R}^n)$. It follows from a classical result that $\|R_j h\|_p \le A_p \|h\|_p$ if $1 < p < \infty$. (See Stein [9].) When $p = \infty$, $R_j h$ does not necessarily belong to $L^\infty(\mathbb{R}^n)$. In fact, it is relatively simple to show that there exists a function $h \in L^\infty(K)$ such that $R_j h \notin L^\infty(\mathbb{R}^n)$ for all $j = 1, 2, \cdots, n$. The main purpose of this paper is to investigate whether or not $H^\infty(K)$ is always *nontrivial; i.e.*, $H^\infty(K) \neq \{0\}$. We remark that $H^\infty(K)$ is a Banach space under the norm $\|h\| = \|h\|_\infty + \sum_{j=1}^n \|R_j h\|_\infty$. Related to $H^\infty(K)$ is the set $\mathscr{H}(K)$ of bounded harmonic functions defined on $\mathbb{R}^{n+1} \setminus K$ and satisfying a Lipschitz condition. If $\mathscr{H}(K)$ consists only of the constants, the set K is called *removable* for harmonic functions satisfying a Lipschitz condition. It turns out that K is removable if and only if $H^\infty(K)$ is trivial (see Theorem 1). We should mention here the related work of Harvey and Polking [6], where they have found sufficient conditions on removable sets for solutions of linear partial differential equations. We remark that the well-known result that $m(K) = 0$ implies K is removable for harmonic functions satisfying a Lipschitz condition, can also be derived from their Theorem 4.3(b).

The problem of removable singularities of harmonic functions satisfying a Lipschitz condition of order α, $0 < \alpha < 1$, has been completely solved by Carleson (see [3, Section VII, Theorem 2]). He proved that K is removable if and only if the $(n - 2 + \alpha)$-dimensional Hausdorff measure $\Lambda_{n-2+\alpha}(K) = 0$.

THEOREM 1. *Let* K *be a compact set of* $\mathbb{R}^n$. *Then* $u \in \mathscr{H}(K)$ *if and only if there exists a function* $h \in H^\infty(K)$ *such that*

$$u(x, y) = \int \log\{(x - t)^2 + y^2\}\, h(t)\, dt + \text{Constant} \qquad \text{if } n = 1$$

and

$$u(x, y) = \int \frac{h(t)}{(|x - t|^2 + y^2)^{(n-1)/2}}\, dt + \text{Constant} \qquad \text{if } n > 1.$$

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