THE RANGES OF ANALYTIC FUNCTIONS WITH CONTINUOUS BOUNDARY VALUES

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Dedicated to my teacher Professor Ivan Vidav on the occasion of his 60th birthday

Denote by Δ, $\overline{\Delta}$, $\partial\Delta$ the open unit disc in $\mathbb{C}$, its closure, and its boundary, respectively. If X is a complex Banach space, we denote by $A(\Delta, X)$ the class of all continuous functions from $\overline{\Delta}$ to X, analytic on Δ, and we write A for $A(\Delta, \mathbb{C})$. We denote the closure and the interior of a set $S \subset X$ by $\overline{S}$ and Int S, respectively. We write $I = \{t: 0 \leq t \leq 1\}$ and denote the set of all positive integers by $\mathbb{N}$.

The main purpose of this note is to present a simple topological description of the sets $f(\overline{\Delta})$, $f \in A$. Note that the topological description of the sets $f(\partial\Delta)$, $f \in A$ is known [2].

We obtain our description by combining some ideas of Pełczyński [9] with some ideas from [6].

Definition. Let P be a subset of a metric space and let $\varepsilon > 0$. We call a finite set $S_\varepsilon \subset P$ an ε-*path-net* for P if given any $x \in P$ there exist $y \in S_\varepsilon$ and a path in P joining x and y whose diameter is less than ε. We say that P is *totally path-connected* if

(i) P is path-connected;

(ii) for every $\varepsilon > 0$ there exists an ε-path-net for P.

Remark. If P is an open subset of a Banach space, then (ii) above is equivalent to the assumption that P has "property S" [7, 12]. Note that there are bounded domains in $\mathbb{C}$ which are not totally path-connected.

THEOREM 1. *Let a subset* K *of* $\mathbb{C}$ *consist of more than one point. Then* $K = f(\overline{\Delta})$ *for some* $f \in A$ *if and only if*

(i) $K = \overline{\text{Int } K}$;

(ii) Int K *is totally path-connected.*

LEMMA 1. *Let* $F \subset \partial\Delta$ *be a closed set of Lebesgue measure* 0, *and let* $\lambda \in \partial\Delta - F$. *Assume that* $p: I \to \mathbb{C}$ *is a path satisfying* $p(0) = 0$. *Let* $\varepsilon > 0$, *and let* $U \subset \overline{\Delta}$ *be a neighborhood of* λ. *There exists* $f \in A$ *satisfying*

(i) $f(F) = \{0\}$;

(ii) $f(\lambda) = p(1)$;

(iii) $|f(x)| < \varepsilon$, $z \in \overline{\Delta} - U$;

(iv) $f(\overline{\Delta}) \subset p(I) + \varepsilon\Delta$.

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