SOME CONGRUENCES FOR BINOMIAL COEFFICIENTS

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1. INTRODUCTION

Let p be a prime number. For integers m, and $n > 0$, and $s > 0$, denote by $M_s(m, n)$ the sum $\sum (-1)^{n-i}\binom{n}{i}$ of alternating binomial coefficients, restricted to those i which are congruent to m modulo p^s. Many results about continuous p-adic-valued functions on the p-adic integers come down to statements about the numbers $M_s(m, n)$. The author has shown that

$$(*) \qquad \operatorname{ord} M_s(m, n) \geq [n/p^{s-1}(p-1)] - 1$$

for all m. Here ord r denotes the exponent to which p divides r, and $[\cdot]$ is the integer-part function. The present note gives more precise information about the congruence properties of the $M_s(m, n)$ modulo powers of p. It is shown that, for fixed m and s, there is equality in (*) for infinitely many n; more specifically,

(i) *if* $k \geq 1$ *and* $n > kp^{s-1}(p-1) + p^{s-1} - 1$, *then* $\operatorname{ord} M_s(m, n) > k - 1$ *for all* m;

(ii) *if* $k \geq 1$, *then* $M_s(m, kp^{s-1}(p-1) + p^{s-1} - 1) \equiv (-p)^{k-1} \pmod{p^k}$ *for every* m.

These results are then applied to obtain a new characterization of uniformly Lipschitz p-adic-valued functions.

2. PRELIMINARY LEMMAS

In this section, p denotes a fixed prime.

LEMMA 1. *Let the integers* a_{in} *be defined by the identity*

$$\binom{py}{i} = \sum_n a_{in}\binom{y}{n}.$$

If $p \nmid a_{in}$, *then* $i = pn$.

Proof. One has $a_{in} = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j}\binom{pj}{i}$. Fix n and let X be an indeterminate. Then $\sum_{i=0}^{n} a_{in}X^i = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j}\sum_{i=0}^{pj}\binom{pj}{i}X^i = ((X+1)^p - 1)^n$. Comparing coefficients yields the lemma.

LEMMA 2. *If* $s \geq 2$ *and* $1 \leq i < p^{s-1}(p-1)$, *then* $\sum_{j=1}^{p-1}\binom{p^{s-1}j}{i}$ *is divisible by* p. *If, in addition,* $i < p^{s-2}(p-1)$, *then* $\sum_{j=1}^{p-1}\binom{p^{s-1}j}{i}$ *is divisible by* p^2.

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