A CLASS OF PURE SUBNORMAL OPERATORS

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Preliminaries. If T is a bounded operator on a Hilbert space $\mathscr{H}$, its spectrum and point spectrum will be denoted by $\sigma(T)$ and $\sigma_p(T)$, respectively. An operator S on $\mathscr{H}$ is *subnormal* if there exists a normal operator N on a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ such that N agrees with S on $\mathscr{H}$. That is, N leaves $\mathscr{H}$ invariant and N restricted to $\mathscr{H}$ is S. N is called the *minimal normal extension* of S if $\mathscr{K}$ is the smallest reducing subspace for N containing $\mathscr{H}$. S is said to be *pure subnormal* if there exists no nontrivial subspace of $\mathscr{H}$ which reduces S and on which S is normal. It is easy to see that the point spectrum of a pure subnormal operator is empty. For further properties of subnormal operators, consult P. Halmos [6] and J. Bram [1].

If K is a compact set in the plane, C(K) denotes the set of (complex-valued) continuous functions on K and R(K) represents the uniform closure of the set of rational functions with poles off K. If F is a closed subset of K and there exists a function f in R(K) such that $f(z) = 1$ for z in F and $|f(z)| < 1$ for z in $K \backslash F$, then F is called a *peak set* for R(K). A peak set consisting of a single point is called a *peak point*. For a detailed exposition of these ideas, consult T. Gamelin [5].

Let S be a pure subnormal operator with minimal normal extension N. If z is any boundary point of a component of the complement of $\sigma(S)$, then $\sigma_p(N) \cap \{z\} = \emptyset$. (See [4, p. 34].) This was generalized by C. Putnam [9, p. 9] in the following way: If Ω denotes the set of peak points of $R(\sigma(S))$ then $\Omega \cap \sigma_p(N) = \emptyset$. For another related result, consult M. Radjabalipour [10, p. 388].

It was asked in both papers, [4, p. 95] and [9, p. 10], whether a stronger result is true. That is, if S and N are as above, is it true that $\sigma_p(N) \cap \partial\sigma(S) = \emptyset$? (Here ∂ denotes the boundary.) The purpose of this paper is to provide a large class of pure subnormal operators S for which $\sigma_p(N) \cap \partial\sigma(S) \neq \emptyset$.

We close this section by mentioning the well-known fact that the minimal normal extension of a pure subnormal operator can have a nonempty point spectrum. The first example of this is due to J. Wermer. He exhibited [12, Theorems 1 and 2] a pure subnormal operator S such that eigenvectors of N span $\mathscr{K}$. However, the eigenvalues of N in this example all lie in the interior of $\sigma(S)$. Another example of this phenomenon was given by D. Sarason. (See [6], Problem 156 and its solution.) The methods of this paper will provide another proof to Sarason's example.

THEOREM. *Let* K *be a compact set in the plane which contains a point* x *which is not a peak point for* R(K). *Then there exists a pure subnormal operator* S *with minimal normal extension* N *such that*

(i) $\sigma(S) \subset K$,

and

(ii) $\sigma_p(N) \cap \sigma(S) \supset \{x\}$.

Received March 12, 1976.

Michigan Math. J. 24 (1977).