ADJOINT REPRESENTATIONS OF FACTOR GROUPS

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Let A be a von Neumann algebra which is a factor, and let G denote either the unitary group of A or the group of invertible members of A. The adjoint representation ϕ of G is defined by $\phi(U)T = UTU^{-1}$ for all $U \in G$, $T \in A$. Below we determine all norm-closed subspaces of A which are invariant under the action of G.

If P, Q are projections in A, we write $P \precsim Q$ if there is a partial isometry $V \in A$ such that $V^*V = P$ and $VV^* \leq Q$. We write $P \sim Q$ if $P \precsim Q$ and $Q \precsim P$; that is, if there is a partial isometry $V \in A$ such that $V^*V = P$ and $VV^* = Q$. We write $P \prec Q$ if $P \precsim Q$ and $P \not\sim Q$. If A is a factor, then for any projections $P, Q \in A$, either $P \precsim Q$ or $Q \precsim P$ [2, p. 218].

If $T \in A$, let R(T), the range projection of T, be the least projection $P \in A$ satisfying $PT = T$. Let S(T) be the subspace spanned by $\{UTU^{-1}: U \in G\}$ and let CS(T) be the norm closure of S(T).

We now recall the ideal structure of a factor. Throughout this paper, *ideal* means norm-closed ideal. If A is a finite factor, then A has no proper ideals [2, p. 257]. If A is of infinite type, then the set of ideals of A is well-ordered by set inclusion [7, 8]. If A is type I_∞ or type II_∞, then the *minimal ideal* of A, hereafter denoted K, is the uniform closure of $\{T \in A: R(T) \text{ is a finite projection}\}$; the members of K are called *compact operators*. If A is type I_∞ or type II_∞, then K is the unique minimal proper ideal of A. If A is of infinite type, the unique maximal proper ideal of A is denoted by J; if A is type III and simple, we define J by $J = \{0\}$.

We denote the identity operator by I. If A is semifinite, we denote by tr the trace on A.

THEOREM 1. *Let* A *be a finite von Neumann algebra factor, let* G *be either the unitary group of* A *or the group of invertible members of* A, *and let* ϕ *be the representation of* G *on* A *defined by* $\phi(U)T = UTU^{-1}$ *for all* $U \in G$, $T \in A$. *Then* G *acts irreducibly on* $\{\lambda I: \lambda \in \mathbb{C}\}$ *and on* $\{T \in A: tr(T) = 0\}$. *Furthermore, these are the only proper invariant subspaces for the action of* G *on* A.

THEOREM 2. *Let* A *be an infinite von Neumann algebra factor, let* G *be either the unitary group of* A *or the group of invertible members of* A, *and let* ϕ *be the representation of* G *on* A *defined by* $\phi(U)T = UTU^{-1}$ *for all* $U \in G$ *and* $T \in A$. *The proper invariant subspaces for the action of* G *on* A *are precisely* $\{\lambda I: \lambda \in \mathbb{C}\}$, *the proper ideals of* A, *and for each proper ideal* $\mathcal{J}$ *of* A, $\{\lambda I + T: \lambda \in \mathbb{C} \text{ and } T \in \mathcal{J}\}$. *The nontrivial irreducible subspaces for the action of* G *on* A *are* $\{\lambda I: \lambda \in \mathbb{C}\}$ *and* (*if* A *is not simple*) *the minimal proper ideal of* A.

If A is a factor of type I or type II, there is associated with A the Hilbert space $L^2(A)$ (see [5]). $L^2(A)$ is the completion of the pre-Hilbert space $\{T \in A: tr(T^*T) < \infty\}$ with respect to the inner product $\langle S, T \rangle = tr(T^*S)$. $L^2(A)$ is a *-algebra, and A acts in a natural manner on $L^2(A)$.

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