FREE INVOLUTIONS ON COMPLEX PROJECTIVE SPACES

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0. INTRODUCTION

Let $h\mathbb{C}P^N$ denote a compact manifold of the same homotopy type as complex projective space $\mathbb{C}P^N$. An easy consequence of the Lefschetz fixed point theorem is: the only group which can act freely on $h\mathbb{C}P^N$ is the cyclic group $\mathbb{Z}/2\mathbb{Z}$. Moreover, in this case it also follows that N must be odd. The purpose of this paper is to classify all free (PL and TOP) $\mathbb{Z}/2\mathbb{Z}$-actions on $h\mathbb{C}P^{2n+1}$. (The existence of a free smooth involution on any $h\mathbb{C}P^3$ was proven by Petrie [2]. His results are implied by Corollary B, below, and the fact that $\pi_i(PL/O) = 0$ for $i < 7$.) We will give invariants which detect the existence of a free involution on $h\mathbb{C}P^{2n+1}$ and determine the structure of the set of equivalence classes of free $\mathbb{Z}/2\mathbb{Z}$-actions on $h\mathbb{C}P^{2n+1}$. In particular, we show that there exist exactly 2^n distinct free PL involutions on $\mathbb{C}P^{2n+1}$. We assume familiarity with the surgery exact sequence [7]:

$$L_{n+1}^s(\pi, w) \xrightarrow{\omega} \mathscr{S}_H^s(M) \xrightarrow{\theta} [M, G/H] \xrightarrow{\sigma} L_n^s(\pi, w),$$

where H = PL or TOP, M is a PL or TOP n-manifold, $\pi = \pi_1(M)$, and $w = w_1(M)$. $\mathscr{S}_H^s(M)$ denotes the set of simple homotopy structures on M.

1. STATEMENT OF RESULTS

In this section, we explain our main results in the topological category. The minor modifications necessary for extending these results to the PL case are given in Section 8.

We write Free Inv$(h\mathbb{C}P^{2n+1})$ for the set of conjugacy classes of free involutions on $h\mathbb{C}P^{2n+1}$. Note, for example, that Free Inv$(\mathbb{C}P^{2n+1}) \neq \emptyset$. In terms of homogeneous coordinates, we can easily describe an element of this set:

$$T[z_0 : z_1 : \cdots : z_{2n+1}] = [-\bar{z}_1 : \bar{z}_0 : -\bar{z}_3 : \bar{z}_2 : \cdots].$$

(In fact, this is the free involution on $\mathbb{C}P^{2n+1}$ induced by the antipodal map in the fibres of the fibration $S^2 \to \mathbb{C}P^{2n+1} \to \mathbb{H}P^n$, where $\mathbb{H}P^n$ denotes quaternionic projective space.) Henceforth, T will denote the above involution and $X = X^{4n+2}$ will denote the orbit space $\mathbb{C}P^{2n+1}/\langle T \rangle$. Let π: $\mathbb{C}P^{2n+1} \to X$ be the natural projection.

Now suppose there exists a free (TOP) involution S on $h\mathbb{C}P^{2n+1}$. Let η denote the $\mathbb{Z}/2\mathbb{Z}$-bundle ρ: $h\mathbb{C}P^{2n+1} \to Y$, where $Y = Y^{4n+2}$ is the orbit space $h\mathbb{C}P^{2n+1}/\langle S \rangle$ and ρ is the natural projection. Define $\mathscr{S}_{TOP}(Y)^P$ to be the set of

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