FOURIER-STIELTJES TRANSFORMS OF STRONGLY CONTINUOUS MEASURES

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1. INTRODUCTION AND PRELIMINARIES

Let G denote throughout a compact abelian group and Γ its dual. For μ belonging to M(G), the space of regular bounded Borel measures on G, the Fourier-Stieltjes transform of μ is given by $\hat{\mu}(\gamma) = \int_G \overline{\gamma(g)}\, d\mu(g)$, $\gamma \in \Gamma$.

Let $0 < \varepsilon < 1$. We shall be concerned with measures μ whose transforms satisfy the following separation condition:

(1, ε) For every $\gamma \in \Gamma$, either $|\hat{\mu}(\gamma)| \geq 1$ or $|\hat{\mu}(\gamma)| < \varepsilon$.

We shall call μ *strongly continuous* if

(2) $|\mu|(g + H) = 0$ for all $g \in G$ and all closed subgroups H of G such that G/H is infinite.

The main result of this paper may be stated qualitatively as follows. For each $C > 0$, there is some $\varepsilon = \varepsilon(C) > 0$ such that if μ satisfies (1, ε), (2), and $\|\mu\| \leq C$, then $\Lambda = \{\gamma \in \Gamma: |\hat{\mu}(\gamma)| \geq 1\}$ is a finite set. It will turn out that $\varepsilon = \varepsilon(C)$ may be chosen independently of G. An alternative formulation of our main result is the following. There is some constant A independent of G such that $\|\mu\| \geq -A(\log \varepsilon)^{1/5}$ for all $\mu \in M(G)$ satisfying (1, ε) and (2), if Λ is infinite.

Previous versions of this theorem include de Leeuw's and Katznelson's [2, p. 221] for the case G = T (the circle group). Ramsey [5] has proved the theorem for those Γ whose torsion subgroup is finite. He also obtained quantitative bounds on the size of Λ. In the general case treated here, such bounds do not exist. To see that, let G be the familiar Cantor group $\prod (\mathbb{Z}/2\mathbb{Z})$, and let Λ be a finite subgroup of Γ of order 2^n. Define μ to be the trigonometric polynomial $\sum_{\gamma \in \Lambda} \gamma(g)$ on G. Since μ is the normalized Haar measure on the compact subgroup $\Lambda^{\perp}$ of G, we have that $\|\mu\| = 1$. It is clear that μ satisfies (1, ε) for every $\varepsilon > 0$ and (2); however, the order of Λ can be arbitrarily large.

An alternate expression of (2) is possible using the canonical homomorphism ϕ of G onto G/H. Define $\phi(\mu)$ to be that measure in M(G/H) determined by the equation $\phi(\mu)(B) = \mu(\phi^{-1}(B))$ for all Borel subsets B of G/H. The equation

$$\int_{G/H} f\, d(\phi(\mu)) = \int_G f \circ \phi\, d\mu$$

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