LUMER'S HARDY SPACES

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In the present paper, the term *pluriharmonic* will always refer to real-valued functions. A pluriharmonic function is thus one whose domain is an open subset Ω of $\mathbb{C}^n$ and which is locally the real part of a holomorphic function.

We define $(LH)^p(\Omega)$ to be the class of all holomorphic functions $f\colon \Omega \to \mathbb{C}$ such that $|f|^p \leq u$ for some pluriharmonic u. (Here $0 < p < \infty$.) This is Lumer's definition of H^p-spaces [1]. When $n = 1$, pluriharmonic is the same as harmonic, so that this definition coincides with the old one ([2], [3]) which involves harmonic majorants of $|f|^p$. But when $n > 1$, then $(LH)^p(\Omega)$ is a proper subclass of what is usually called $H^p(\Omega)$. (See, for example, [6].)

The use of pluriharmonic majorants leads to some appealing properties of $(LH)^p(\Omega)$. For example, holomorphic invariance is a triviality: if Φ is a holomorphic map of Ω_1 into Ω_2 and if $f \in (LH)^p(\Omega_2)$, then obviously $f \circ \Phi \in (LH)^p(\Omega_1)$.

To see another example, let Ω be simply connected. If $f \in (LH)^p(\Omega)$ for some $p \in (0, \infty)$, then $\log |f| \leq \Re g$ for some holomorphic g in Ω. Setting $h = f \cdot \exp(-g)$, it follows that $|h| \leq 1$. *Thus every* $f \in (LH)^p(\Omega)$ *has the same zeros as some* $h \in H^\infty(\Omega)$. This is in strong contrast to what is known [4] about zero sets of the usual H^p-functions in the unit ball or the unit polydisc of $\mathbb{C}^n$.

However, from the standpoint of functional analysis, the $(LH)^p$-spaces have unexpectedly pathological properties. The purpose of the present paper is to describe some of these for the case $\Omega = B$, the open unit ball of $\mathbb{C}^n$; from now on, $n > 1$.

When $1 \leq p < \infty$, $(LH)^p(B)$ can be normed by defining

$$\|f\|_p = \inf u(0)^{1/p}, \tag{1}$$

the infimum being taken over all pluriharmonic majorants u of $|f|^p$ in B. As pointed out in [1], this norm turns $(LH)^p(B)$ into a Banach space.

For $0 \leq r < 1$, we use the notation f_r to denote the function defined for $z \in B$ by $f_r(z) = f(rz)$.

We let $\mathscr{U}$ denote the (compact topological) group of all unitary transformations of $\mathbb{C}^n$. Clearly, every $U \in \mathscr{U}$ maps B onto B.

As usual ℓ^∞ is the Banach space of all bounded complex sequences, and c_0 is the subspace of ℓ^∞ consisting of those sequences that converge to 0.

Here is our main result:

THEOREM. *Fix* p, $1 \leq p < \infty$, *and fix* $\varepsilon > 0$.

(i) *There exists a linear map of* ℓ^∞ *into* $(LH)^p(B)$ *which assigns to each* $\gamma \in \ell^\infty$ *a function* f_γ *that satisfies* $\|\gamma\|_\infty \leq \|f_\gamma\|_p \leq \|f_\gamma\|_\infty \leq (1+\varepsilon)\|\gamma\|_\infty$.

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