OPERATOR ALGEBRAS LEAVING COMPACT OPERATOR RANGES INVARIANT

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We show that a closed operator whose domain is the range of a compact operator has closed range. This has two easy consequences: a closed operator whose range is contained in that of a compact operator must itself be compact, and a transitive algebra whose only proper invariant operator ranges are the ranges of compact operators must be strongly dense.

We consider operators from any Banach space into any other Banach space. (In all the results other than Theorem 2, the spaces can be real or complex; for Theorem 2, the space must be complex.) A *closed operator* is a linear transformation T with domain $\mathscr{D}(T)$ in some space $\mathscr{X}$ and range in some $\mathscr{Y}$ such that $\{x \oplus Tx\colon x \in \mathscr{D}(T)\}$ is a closed subspace of $\mathscr{X} \oplus \mathscr{Y}$, where the norm on $\mathscr{X} \oplus \mathscr{Y}$ is any which is equivalent to the norm defined by $\|x \oplus y\| = \|x\| + \|y\|$. Then the projections of $\mathscr{X} \oplus \mathscr{Y}$ onto $\mathscr{X}$ and onto $\mathscr{Y}$ are bounded operators.

We begin with a well-known fact which follows, for example, from Douglas [1].

LEMMA 1. *A bounded operator whose range is contained in the range of a compact operator is itself compact.*

Proof. (Similar to [1].) Let K be the compact operator, with domain $\mathscr{Y}$. By replacing K by its natural quotient on $\mathscr{Y}/\ker K$ if necessary, we can assume that K is injective. Now if T is a bounded operator with $\operatorname{ran} T \subset \operatorname{ran} K$, then $K^{-1}T$ is a closed operator whose domain is a Banach space, so $K^{-1}T = C$ is bounded. Hence, $T = KC$ is compact.

Theorem 1 below generalizes Lemma 1 to closed operators.

LEMMA 2. *A closed operator whose domain is contained in the range of a compact operator must have closed range.*

Proof. Let T be the closed operator; by going to a quotient space if necessary, we can assume that T is injective. Suppose that the domain $\mathscr{D}(T)$ is contained in the range of the compact operator K_0. Then the projection K of the graph of T onto the first coordinate space is a bounded operator with $\operatorname{ran} K \subset \operatorname{ran} K_0$, so K is compact by Lemma 1. Clearly, K is injective and has range equal to $\mathscr{D}(T)$.

Now TK is a closed operator with domain a Banach space $\mathscr{D}(K)$, so TK is a bounded operator C. Clearly,

$$\{x \oplus Tx\colon x \in \mathscr{D}(T)\} = \{Kz \oplus Cz\colon z \in \mathscr{D}(K)\}.$$

Thus to show ran T is closed, it suffices to show ran C is closed. Define the mapping U by $Uz = Kz \oplus Cz$ for $z \in \mathscr{D}(K)$. Then U is a bounded injective operator mapping the Banach space $\mathscr{D}(K)$ onto the graph of T, so U is bounded below. We must prove that C is bounded below.

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