COMPACT FAMILIES OF UNIVALENT FUNCTIONS

Eric P. Kronstadt

Let D be a proper domain in the complex plane $\mathbb{C}$, H(D) the space of holomorphic functions on D, and $H_u(D)$ the subset of univalent functions in H(D). We endow H(D) with the topology of uniform convergence on compact sets. If $L = (\ell_1, \ell_2, \cdots, \ell_n)$ is an n-tuple of continuous, linearly independent, linear functionals on H(D), and $Q = (q_1, q_2, \cdots, q_n) \in \mathbb{C}^n$, define

$$\mathscr{F}(D, L, Q) = \{f \in H_u(D): L(f) = Q\}.$$

In [1], Hengartner and Schober proved

THEOREM A. *If* $\mathscr{F} = \mathscr{F}(D, (\ell_1, \ell_2), (q_1, q_2))$ *is nonempty, and* (ℓ_1, ℓ_2) *satisfies*

$$(*) \qquad \ell_1(1)\,\ell_2(g) \neq \ell_2(1)\,\ell_1(g), \qquad \textit{for every } g \in H_u(D),$$

then $\mathscr{F}$ *is compact. Moreover, if* D *has a "strongly dense boundary" and* $\mathscr{F}$ *is nonempty and compact, then* (*) *holds.*

This paper is concerned with generalizing Theorem A to the case of more than two linear functionals.

Clearly, if (*) held for one pair of the n linear functionals $\ell_1, \ell_2, \cdots, \ell_n$, then $\mathscr{F}(D, L, Q)$ would be compact whenever it were nonempty. On the other hand, as the following example shows, $\mathscr{F}$ may be compact even if (*) fails for each pair of the n linear functionals.

Example. Let D be the unit disk $\Delta = \{z: |z| < 1\}$; let $\ell_1(f) = f''(0) + f'(0)$, $\ell_2(f) = f(0)$, $\ell_3(f) = f''(0)$; and let $q_1 = 1$, $q_2 = q_3 = 0$. If $I(z) = z$, then $I \in \mathscr{F}(\Delta, L, Q)$; so $\mathscr{F}(\Delta, L, Q)$ is nonempty. Clearly,

$$\mathscr{F}(\Delta, L, Q) = \{f \in H_u(\Delta): f(0) = 0,\ f'(0) = 1\} \cap \{f \in H(\Delta): f''(0) = 0\}.$$

The first set on the right-hand side is well known to be compact, and the second is closed. Therefore, $\mathscr{F}(\Delta, L, Q)$ is nonempty and compact. On the other hand, if $h(z) = z - z^2/2$, then $h \in H_u(\Delta)$, and

$$\begin{aligned} 0 &= \ell_1(1)\,\ell_2(h) = \ell_2(1)\,\ell_1(h) \\ &= \ell_1(1)\,\ell_3(I) = \ell_3(1)\,\ell_1(I) \\ &= \ell_2(1)\,\ell_3(I) = \ell_3(1)\,\ell_2(I). \end{aligned}$$

Thus, (*) fails for each pair of the three linear functionals.

The generalization of Theorem A we wish to explore arises from the following observation. Let Ker(L) denote the kernel of L.

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