ON A UNIQUENESS THEOREM IN CONFORMAL MAPPING

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Let $f(z) = z + b_0 + b_1 z^{-1} + \cdots + b_n z^{-n} + \cdots$ be univalent and holomorphic in $D^* = \{z: |z| > 1\}$, up to the simple pole at ∞. The set $B = \mathbb{C} \setminus f(D^*)$ is a compact continuum whose diameter we denote by d. According to [3, Abschnitt IV, Aufgabe 141, S.25 und S.199], d satisfies the inequalities $2 \leq d \leq 4$. Moreover, d equals 4 if and only if B is a straight segment of length 4, and d equals 2 if and only if B is a disc; that is, if $f(z) = z + b_0$. But a proof that d equals 2 only if $f(z) = z + b_0$ was missing. In 1969, J. A. Jenkins [2] closed this gap with a proof based on the following facts: if $d = 2$, then B is bounded by a rectifiable Jordan curve of length at most 2π; hence, f' belongs to the Hardy class H_1, the function $f(e^{i\theta})$ is absolutely continuous, and $df(e^{i\theta})/d\theta = ie^{i\theta} f'(e^{i\theta})$ almost everywhere. It is the purpose of this note to give an *elementary* proof.

THEOREM. *The diameter* d *of* B *satisfies the inequality* $d \geq 2$, *and equality occurs if and only if* $f(z) = z + b_0$.

Let C_r denote the circle $\{z: |z| = r\}$ oriented in the positive sense. For $r > 1$, let $\Gamma_r = f(C_r)$. Let d_r denote the diameter of Γ_r and L_r the length of Γ_r.

The proof of the inequality $d_r \geq 2$ is elementary (*cf.* [3]). From the relation

$$f(z) - f(-z) = 2z + 2b_1 z^{-1} + 2b_3 z^{-3} + \cdots, \quad |z| > 1,$$

it follows that

$$2 = \left| \frac{1}{2\pi i} \int_{C_r} (f(z) - f(-z)) \frac{dz}{z^2} \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta}) - f(-re^{i\theta})| \frac{d\theta}{r} \leq \frac{1}{r} d_r, \quad r > 1;$$

and this implies $d \geq 2$.

We now give three lemmas that we need for the proof of the uniqueness part of the theorem.

LEMMA 1. *For* $r \geq 1$, $d_r \leq rd$.

Proof. Let z_1 and z_2 be two points on C_r such that $|f(z_1) - f(z_2)| = d_r$, and let $\Phi(z) = f(z_1 z/z_2) - f(z)$. Then, for $\rho > 1$, it follows from the relations $\max_{|z|=\rho} |\Phi(z)| \leq d_\rho$ and $\lim_{\rho \to 1} d_\rho = d$ that

$$\limsup_{|z| \to 1} |\Phi(z)/z| \leq d.$$

By the maximum modulus principle, $|\Phi(z)| \leq |z| d$. Because $\Phi(z_2) = f(z_1) - f(z_2)$, it follows further that $d_r \leq rd$.

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