A NON-NOETHERIAN TWO-DIMENSIONAL HILBERT DOMAIN WITH PRINCIPAL MAXIMAL IDEALS

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All rings considered in this paper are assumed to be commutative and to contain an identity element.

A. V. Geramita (personal communication) has raised the question of whether a Hilbert domain R is Noetherian if each maximal ideal of R is finitely generated. This question arises naturally in at least two contexts. First, the question arises in connection with the well-known theorem of I. S. Cohen to the effect that a ring S is Noetherian if each prime ideal of S is finitely generated [3, Theorem 2]; to wit, O. Goldman introduced the term *Hilbert ring* in [13, p. 136], and his definition of the term was a ring in which each prime ideal is an intersection of maximal ideals. (W. Krull independently considered the class of Hilbert rings in [18]; the terminology of [18, p. 354] for such rings is *Jacobsonsche Ringe.* In different terminology, a Hilbert ring is a ring in which each prime ideal is a J-*radical ideal,* or a J-*prime ideal* [22, p. 631]; for yet another perspective of Hilbert rings, see Section 1-3 of [17].) Second, the property that each of its maximal ideals is finitely generated is inherited by each polynomial ring $R[X_1, \cdots, X_n]$ in finitely many indeterminates over a Hilbert ring R [17, Exercise 8, p. 20]; a straightforward proof of this result can be obtained from the fact that a ring S is a Hilbert ring if and only if $M \cap S$ is a maximal ideal of S for each maximal ideal M of $S[X_1, \cdots, X_n]$ (see [13, Theorem 5] or [18, Section 2]), but an alternate proof would follow at once from the Hilbert Basis Theorem *if* the answer to Geramita's question were affirmative. In Example 1, we construct a Hilbert domain that shows that the answer to Geramita's question is negative. (We use the term *Hilbert domain* to refer to a Hilbert ring that is also an integral domain.) Since a one-dimensional Hilbert domain (or a zero-dimensional Hilbert ring) with finitely generated maximal ideals is Noetherian by Cohen's theorem, such a domain D must have (Krull) dimension at least 2. We show, in fact, that there is a two-dimensional example D_0 that is a Bezout domain (and hence maximal ideals of D_0 are principal) and a subring of Q(X), the rational function field in one variable over the rational field Q. (Examples of one-dimensional, non-Noetherian, Bezout, Hilbert rings with principal maximal ideals are fairly easy to obtain from the well-known $D + M$ construction of [5, Appendix 2]; such rings must contain zero divisors, and a specific example of such a ring is mentioned in the paragraph following Example 1.)

Throughout the remainder of the paper, we use the following notation. Let D be a Dedekind domain with quotient field K, and for each element α in A, an infinite set, let E_α be an infinite family of maximal ideals of D, where $E_\alpha \cap E_\beta = \emptyset$ if α and β are distinct elements of A. Let $\{d_\alpha\}_{\alpha \in A}$ be a subset of D such that $d_\alpha \neq d_\beta$ for $\alpha \neq \beta$, and for each α in A, let $V_\alpha = K[X]_{(X-d_\alpha)}$; thus, V_α is a rank-one discrete valuation ring of the form $K + M_\alpha$, where $M_\alpha = (X - d_\alpha) K[X]_{(X-d_\alpha)}$

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