EXTRINSIC SPHERES IN KÄHLER MANIFOLDS

Bang-yen Chen

1. INTRODUCTION

An n-dimensional submanifold M^n of an arbitrary Riemannian manifold $\tilde{M}^m$ is called an *extrinsic sphere* if it is umbilical and has parallel mean curvature vector $H \neq 0$ [4]. (Dimensions of manifolds are real dimensions.) We say that a Riemannian manifold $\tilde{M}^m$ is *sufficiently curved* if for every point $x \in \tilde{M}^m$, the maximal linear subspace V of the tangent space $T_x(\tilde{M}^m)$ of $\tilde{M}^m$ at x with $\tilde{R}(X, Y) = 0$ for $X, Y \in V$ has dimension less than $m - 2$, where $\tilde{R}$ denotes the curvature tensor of $\tilde{M}^m$.

In this paper, we shall study extrinsic spheres in an arbitrary Kähler manifold. In particular, we shall prove the following.

THEOREM 2. *There exists no complete orientable extrinsic sphere of codimension two in any sufficiently curved Kähler manifold.*

Remark 1. A standard $(m - 1)$-sphere $(m \geq 3)$ of small radius can be imbedded as an extrinsic sphere in the complex projective space $P^{2m}(\mathbb{C})$, which is positively curved by the Fubini-Study metric [3]. For the classification of umbilical submanifolds in complex space forms, see [3]. For the nonexistence of extrinsic spheres of codimension two in irreducible Hermitian symmetric spaces of dimension greater than 2, see [2].

2. PRELIMINARIES

Let M^n be an n-dimensional submanifold of a 2m-dimensional Kähler manifold $\tilde{M}^{2m}$ with complex structure J and Kähler metric g, and let ∇ and $\tilde{\nabla}$ be the covariant differentiations on M^n and $\tilde{M}^{2m}$, respectively. Then the second fundamental form σ is defined by $\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$, where X and Y are vector fields tangent to M^n and σ is a normal-bundle-valued symmetric 2-form on M^n. For a vector field ξ normal to M^n, we write

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where $-A_\xi X$ (respectively, $D_X \xi$) denotes the tangential component (respectively, the normal component) of $\tilde{\nabla}_X \xi$. A normal vector field ξ is said to be *parallel* if $D\xi = 0$. The submanifold is said to be *umbilical* if $\sigma(X, Y) = g(X, Y)H$, where $H = (\text{trace } \sigma)/n$ is the *mean curvature vector* of M^n in $\tilde{M}^{2m}$.

Let R, $\tilde{R}$, and R^N be the curvature tensors associated with ∇, $\tilde{\nabla}$, and D, respectively. For example, $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$.

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