THE GENUS OF A CLOSED SIMPLY CONNECTED MANIFOLD

Peter Andrews

Let NH be the homotopy category of nilpotent CW-complexes [8]. If $X \in NH$ has finite type (*i.e.*, all its homotopy groups are finitely generated), then define the genus $G(X)$ of X to be the collection of all objects of finite type $Y \in NH$ such that Y_p is homotopy equivalent to X_p for all primes p. Here X_p is the p-localization of X [8, 15]. A homotopy-theoretic property is said to be *generic* if it is shared by all or none of the members of a genus.

In [8], Hilton, Mislin, and Roitberg prove that "being a Poincaré duality space" and "being S-reducible" are both generic properties. This leads them to ask whether "having the homotopy type of a closed manifold" and "having the homotopy type of a closed π-manifold" are generic properties. This paper gives a partial answer to these questions in the simply connected case. The main results are:

THEOREM A. *Let* M^m *be a closed, simply connected, piecewise linear (topological) manifold of dimension* $m \geq 5$, *and let* $X \in G(M)$. *Then* X *is the homotopy type of a closed, piecewise linear (topological) manifold.*

THEOREM B. *Let* M^m *be a closed, simply connected, smooth* (C^∞) *manifold, with* $m \geq 5$ *and* m *odd, and let* $X \in G(M)$. *Then* X *is the homotopy type of a closed smooth manifold.*

THEOREM C. *There exists a closed, simply connected, smooth manifold* B^8 *and a homotopy type* $X \in G(B)$ *such that* X *is not the homotopy type of a closed smooth manifold.*

THEOREM D. *Let* M^m *be a closed, simply connected, smooth* π*-manifold with* $m \geq 5$ *and* $m \neq (2^i - 2)$, *and let* $X \in G(M)$. *Then* X *is the homotopy type of a closed smooth* π*-manifold.*

The proofs of Theorems A, B, and D follow roughly the same plan. A space in the genus of a closed manifold is shown to be a Poincaré duality space whose Spivak normal fibration can be given the structure of an R^m-bundle. This reduces the theorems to a problem of calculating surgery obstructions. The calculation is quite easy in the cases of Theorems A and B, and in that part of Theorem D when $m \not\equiv 2 \pmod 4$. When $m \equiv 2 \pmod 4$, Brown's version of the Kervaire invariant is used to examine the appropriate obstruction.

Section 1 carries out the first part of the program by making minor modifications in the techniques developed by Peter Kahn [10] and independently by the author [1] to examine the mixing of homotopy types of manifolds (see [8, Section II.7] for definitions). In genus questions, one is looking at a space whose p-localizations all agree with the p-localizations of a given space. In mixing questions, one is looking at a space whose localizations agree with those of one space for a given set of primes, and with those of a second space for the complementary set of primes. The similarity of the theorems about mixing in [10] and [1], and those about the genus in this paper, reflects the similarity of these situations. The analysis of the Kervaire

Received January 7, 1976. Revision received October 11, 1976.

Michigan Math. J. 23 (1976).