BRANCHED COVERINGS AND ORBIT MAPS

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1. INTRODUCTION

Let $f: X \to Y$ be a finite-to-one, closed and open (continuous) map, and consider the question as to when the induced homomorphism in rational (sheaf-theoretic) cohomology

(1.1) $$f^*: H^*(Y; \mathbb{Q}) \to H^*(X; \mathbb{Q}) \text{ is injective.}$$

Questions of this sort seem to have originated in a dual form with Eilenberg [6] and Whyburn [11]. See also [2]. If X and Y are connected orientable manifolds, then a standard Poincaré duality and degree argument shows that (1.1) is true. Also, if f happens to be the orbit map for a finite group action on X, then (1.1) again holds because of the existence of a transfer map in this context [1; II.19].

On the other hand, (1.1) is known to be false in general, since Bredon [2] and others have constructed finite-to-one, open, piecewise linear maps from compact contractible polyhedra onto the 2-sphere.

In this paper, X and Y shall always be assumed to be locally connected Hausdorff spaces. Additional hypotheses on the map f or on the spaces X and Y are then considered which guarantee that (1.1) holds. In Section 2, standard geometric notions of degree and local degree for f are defined and it is shown that if the degree of f is always equal to the sum of the local degrees of f on each point inverse, then (1.1) holds. The proof involves the construction of transfer homomorphisms. Such a map f can be viewed as a generalization of Fox's notion [7] of a branched covering. In Section 3, the concept of a (topological) normal n-circuit is introduced, generalizing that of an n-manifold. It is then shown that if X is a normal n-circuit, then Y is also a normal n-circuit and the degree of f is always equal to the sum of the local degrees in each point inverse of f. Proofs here are based on work of Černavskiĭ [3], [4] and Väisälä [10]. Finally, in Section 4 the automorphism group of f consisting of all homeomorphisms $g: X \to X$ such that $fg = f$ is calculated when the branch set of f does not locally separate X. This result is then used to characterize those maps f which can be identified with orbit maps for finite group actions in the case where X is a simply connected piecewise linear manifold, Y is a polyhedron, and f is a piecewise linear map.

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