FRAMED MANIFOLDS WITH A FIXED POINT FREE INVOLUTION

Edgar H. Brown, Jr.

The aim of this note is to prove that every framed cobordism class of positive dimension can be represented by a framed manifold with a fixed point free involution which preserves the framing. In the following, all manifolds and maps are smooth wherever this makes sense.

Suppose M is a closed, compact m-manifold with a fixed point free involution t, ν_M is the normal bundle of $M \subset R^{m+k}$ (k large), and f: $\nu_M \to R^k$ is a framing. We say that t *preserves* f if the following condition is satisfied. Let N = M/t and let p: M $\to$ N be the projection. Then $M \to N \subset R^{m+k}$ is an immersion and hence p is covered by a canonical map s: $\nu_M \to \nu_N$ which is unique up to homotopy. We say that t preserves f if f = gs, where g: $\nu_N \to R^k$ is a framing. Let (M, f)/t = (N, g). Let Ω_*^{fr} denote the framed cobordism group and ${}_2\Omega_*^{fr}$ its two-primary part.

We prove the following theorem.

THEOREM 1. *If* $\alpha \in \Omega_m^{fr}$ (m > 0), *then* α *can be represented by* (M, f), *where* M *admits a fixed point free involution* t *which preserves* f. *If* $\alpha \neq 0$ *and* $\alpha \in {}_2\Omega_m^{fr}$, *then* (M, f) *and* t *can be chosen so that* M *is* [m/2]*-connected and* (M, f)/t *is framed cobordant to zero.*

We begin the proof of Theorem 1 by stating and proving a result of N. Ray [2]. Let P^{k-1} be real projective (k - 1)-space, let A: $R^k \to R^k$ be given by

$$A(x_1, x_2, \cdots, x_k) = (-x_1, x_2, \cdots, x_k),$$

and let λ: $P^{k-1} \to SO_k$ be the composition of A and the map which assigns to each line ℓ, the reflection through the orthogonal complement of ℓ. If g: $\nu_N \to R^k$ is a framing and u: N $\to P^{k-1}$, let ug: $\nu_N \to R^k$ be the framing given by

$$ug(v) = (\lambda up(v))(g(v)),$$

where p: $\nu_N \to N$ is the projection.

THEOREM 2 (N. Ray). *If* $\alpha \in {}_2\Omega_m^{fr}$ (m > 0, $\alpha \neq 0$), *then* α *can be represented by* (N, ug), *where* (N, g) *is framed cobordant to zero and* u_*: $\pi_i(N) \to \pi_i(P^{k-1})$ *is an isomorphism for* 2i < m.

Proof. Let $T(\nu_N)$ be the Thom space of ν_N, that is, the disc bundle modulo the sphere bundle, and let t: $S^{m+k} \to T(\nu_N)$ be the Thom-Pontrjagin construction. We identify Ω_m^{fr} with $\pi_{m+k}(S^k)$ under the map $\{N, g\} \to [T(g)t]$.

Let D^k be the unit k-disc and $S^{k-1} \circ P^{k-1}$ be $D^k \times P^{k-1}$ modulo the relation $(x, y) \approx (x, y')$ for $x \in S^{k-1}$. Let J: $S^{k-1} \circ P^{k-1} \to S^k = D^k/S^{k-1}$ be given by $J(x, y) = \lambda(y)(x)$. D. S. Kahn and S. B. Priddy [1] have shown that

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