COMPACTIFICATION OF COVERING SPACES OF COMPACT 3-MANIFOLDS

Jonathan Simon

The problem of determining conditions under which a noncompact 3-manifold $\tilde{M}$ permits a *manifold compactification*, that is, an embedding $h: \tilde{M} \to Q$, where Q is compact and $h(\text{int } \tilde{M}) = \text{int } Q$, has recently received considerable attention (see [6], [2], [15], and references therein). Since h is a homotopy equivalence, $\tilde{M}$ must have a legal homotopy type: if $\tilde{M}$ contains no 2-sided projective plane, then, by Theorem 2.1 of [14] and Theorem 3.2 of [4], the assumption that

$$\pi_1(\tilde{M}) \quad \text{and} \quad \text{image}(\pi_2(\tilde{M}) \to H_2(\tilde{M}; \mathbb{Z}))$$

are finitely generated is sufficient to guarantee that $\tilde{M}$ has the homotopy type of a compact 3-manifold. Also, Kneser's theorem [11] and the unresolved Poincaré conjecture force us to require that $\tilde{M}$ contain at most finitely many fake 3-cells. But these conditions are far from sufficient: there exist contractible manifolds $\tilde{M} \subseteq \mathbb{R}^3$ that are open [13] or have interior $\mathbb{R}^3$ and boundary $\mathbb{R}^2$ [6, p. 230] but have no manifold compactifications. The obstruction, a sort of homotopic wildness at infinity, is studied in [6], [2], and [15].

This paper considers the question of the existence of a manifold compactification primarily in the case where $\tilde{M}$ is a covering space of a compact 3-manifold. Postulating P^2-irreducibility to eliminate problems with π_2, projective planes, and fake 3-cells, and hoping that the regularities inherent even in irregular covering spaces prevent wildness of π_1 at infinity, we are led to the following conjecture.

If M *is a* P^2*-irreducible, compact, connected* 3*-manifold and* H *is a finitely generated subgroup of* $\pi_1(M)$, *then the covering space* $\tilde{M}(H)$ *of* M *corresponding to* H *has a manifold compactification.*

In Section 2, we consider the technical problems involved in pasting together compactifications of manifolds to obtain a compactification of their union. The compactification theorem developed there is the basis for Section 3, in which we consider the problem of compactifying covering spaces.

In Corollary 3.3, we establish the conjecture on compactifying covering spaces for the cases where M is a line bundle over a 2-manifold or the product of a 2-manifold with S^1. In addition, we show in Corollary 3.2 that the class of manifolds for which the conjecture holds is closed under the operations of pasting along disks or along incompressible annuli, Möbius bands, and tori. In Theorem 3.7, we expand the class of well-behaved manifolds by restricting H: if M is a compact, connected, P^2-irreducible, *sufficiently large* manifold and the intersection of H with each finitely generated subgroup of $\pi_1(M)$ is finitely generated, then $\tilde{M}(H)$ has a manifold compactification. In particular, we indicate in Corollary 3.8 that Theorem 3.7 can be applied whenever H is abelian or is a subgroup of $\text{image}(\pi_1(\partial M) \to \pi_1(M))$.

Received September 24, 1973. Revision received January 26, 1976.
The author was partially supported by NSF Grant GP-29430.

Michigan Math. J. 23 (1976).