SUFFICIENT CONDITIONS FOR RANK-ONE COMMUTATORS AND HYPERINVARIANT SUBSPACES

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Let $\mathscr{X}$ be an infinite dimensional complex Banach space, and let $\mathscr{L}(\mathscr{X})$ denote the algebra of all bounded linear operators on $\mathscr{X}$. In an earlier paper [5] (see also [1]), the authors obtained the following extension of the celebrated theorem of V. Lomonosov [6]:

THEOREM A. *Suppose* T *is an operator in* $\mathscr{L}(\mathscr{X})$ *and there exists a nonzero compact operator* K *in* $\mathscr{L}(\mathscr{X})$ *such that the rank of* TK - KT *is less than or equal to one. Then* T *has a nontrivial hyperinvariant subspace.*

(Recall that a subspace $\mathscr{M}$ of $\mathscr{X}$ is a nontrivial hyperinvariant subspace for an operator T in $\mathscr{L}(\mathscr{X})$ if $(0) \neq \mathscr{M} \neq \mathscr{X}$ and $T'\mathscr{M} \subset \mathscr{M}$ for every operator T' in $\mathscr{L}(\mathscr{X})$ that commutes with T.)

The main purpose of this note is to obtain some results concerning the size of the class of operators to which Theorem A applies. In particular, let $\Delta(\mathscr{X})$ denote the set of all those operators T in $\mathscr{L}(\mathscr{X})$ with the property that there exists a compact operator K such that the rank of TK - KT is *equal to* one. The interest in the class $\Delta(\mathscr{X})$ derives, of course, from Theorem A. It turns out that $\Delta(\mathscr{X})$ is quite large, and in particular, if $\mathscr{X}$ is a separable, infinite dimensional Hilbert space $\mathscr{H}$, we are presently unable to exhibit any nonscalar operator in $\mathscr{L}(\mathscr{H})$ that does not belong to $\Delta(\mathscr{H})$. Thus it is conceivable that the hyperinvariant subspace problem for (separable) Hilbert space can be settled affirmatively by showing that $\Delta(\mathscr{H}) = \mathscr{L}(\mathscr{H}) \setminus \{\lambda\}$.

If $\mathscr{X}$ is, once again, an arbitrary infinite dimensional complex Banach space, and if $f \in \mathscr{X}$ and $\phi \in \mathscr{X}^*$, we shall write $f \otimes \phi$ for the operator of rank one in $\mathscr{L}(\mathscr{X})$ defined as follows: $(f \otimes \phi)(g) = \phi(g)f$, $g \in \mathscr{X}$. Clearly every operator in $\mathscr{L}(\mathscr{X})$ of rank one has the form $f \otimes \phi$ for some choice of nonzero vectors f in $\mathscr{X}$ and ϕ in $\mathscr{X}^*$. Furthermore, for any T in $\mathscr{L}(\mathscr{X})$, an easy calculation shows that

$$T(f \otimes \phi) - (f \otimes \phi)T = (Tf \otimes \phi) - (f \otimes T^*\phi) .$$

This fact will be used several times in what follows. Finally, the spectrum of an operator T in $\mathscr{L}(\mathscr{X})$ will be denoted by $\sigma(T)$.

We begin with the following elementary proposition whose proof we omit.

PROPOSITION 1. *An operator* T *in* $\mathscr{L}(\mathscr{X})$ *belongs to* $\Delta(\mathscr{X})$ *if and only if* $\alpha T + \beta \in \Delta(\mathscr{X})$ *for all scalars* $\alpha \neq 0$ *and* β. *Furthermore, if* $T \in \Delta(\mathscr{X})$ *and if* $S \in \mathscr{L}(\mathscr{X})$ *and is quasisimilar to* T *(that is, if there exist operators* X *and* Y *in* $\mathscr{L}(\mathscr{X})$ *with trivial kernels and cokernels such that* TX = XS, YT = SY), *then* $S \in \Delta(\mathscr{X})$. *Finally, if* $T \in \Delta(\mathscr{X})$, *then* $T^* \in \Delta(\mathscr{X}^*)$.

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