A SPLITTING CONDITION USING BLOCK THEORY

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Let A be a normal abelian subgroup of the finite group G (only finite groups are considered here). There are various conditions which imply that G is split over A. The most celebrated, Schur's splitting theorem, is the case when the index of A in G is relatively prime to the order of A. We present here a block theoretic condition for the existence of a complement. The proof follows closely the elegant non-cohomological proof of Schur's splitting theorem given by Wielandt. In fact, we obtain Schur's splitting theorem as a corollary.

1. MODULES IN THE PRINCIPAL BLOCK

Let F be a splitting field of characteristic p for the finite group N, and let e denote the centrally primitive idempotent of the group algebra F[N] corresponding to the principal p-block of N. Write $e = \sum_{x \in N} c_x x$. We list some properties of e:

(a) Each c_x lies in the prime subfield GF(p) of F;

(b) The function $x \to c_x$ is a class function on N;

(c) $\sum_{x \in N} c_x = 1$.

Statement (a) follows from explicit formulas for the coefficients c_x given by M. Osima [2] and the fact that all the algebraic conjugates of a fixed ordinary irreducible character belonging to the principal p-block also belong to the principal p-block. The second statement is clear, as $e \in Z(F[N])$. Finally, if $s = \sum_{x \in N} x$, then s spans the unique one-dimensional space of invariants of F[N]. Since e acts as the identity on this subspace, se = s. However, sx = s for all $x \in N$, so

$$s\left(\sum_{x \in N} c_x x\right) = cs, \qquad \text{where } c = \sum_{x \in N} c_x.$$

Hence c = 1 and (c) follows.

The three statements about e above will be essentially the only facts needed from representation theory.

If V is an F[N]-module, then by definition V belongs to the principal p-block of N, provided $Ve \doteq V$. Notice that if U is any GF(p)[N]-module, then Ue is defined because of (a) above, and $U \otimes_{GF(p)} F$ lies in the principal p-block of N precisely when Ue = U. Thus, we may define a GF(p)[N]-module U to belong to the principal p-block whenever Ue = U. As the formula for e does not change when F is replaced by any other splitting field, this notion is well defined. Notice that if no composition factor of U belongs to the principal p-block, then Ue = 0.

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