REMARKS ON FLAT MODULES

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0. INTRODUCTION

The property of a commutative ring A that finitely generated flat A-modules be projective is known to be a property of the topology of Spec(A), the set of prime ideals of A [10]. Similarly, the property that pure ideals be generated by idempotents depends only on the topology of Spec(A) [8]. In fact these properties do not depend on the Zariski topology as much as on a weaker topology on Spec(A). In recent years several results have been obtained about the stability of these properties. The purpose of this note is to determine the extent to which these results follow from topological properties.

In Section 1, we show that the above properties are equivalent to very simple conditions on a certain quotient space of Spec(A). In Section 2, we study the extent to which the above properties are inherited from an A-algebra, or by an A-algebra, in terms of these quotient spaces. It becomes apparent that more than just the map induced on the quotient spaces by the structure homomorphism $A \to B$ is involved in this stability. In Section 3, we show that the fact that a polynomial ring or power series ring over A has either of these properties if and only if A does, is due to the fact that the relevant topological spaces are homeomorphic. We conclude with some remarks about the relationship between the property that finitely generated flat A-modules be projective and the property that finite type flat A-algebras be finitely presented.

In this note all rings are commutative with identity. If M is an A-module and $p \in \operatorname{Spec}(A)$, then $rk_M(p)$ denotes the dimension of $M \otimes_A k(p)$ as a vector space over $k(p) = R_p/pR_p$. We will label the conditions we are interested in as in [18] and [19].

(F) Every pure ideal of A is generated by an idempotent.

(f) Every pure ideal of A is generated by idempotents.

A ring A satisfies (F) if and only if every finitely generated flat A-module is projective. The papers [3], [10], [16], [17], and [18] are good references for F-rings. See [8] and [19] for f-rings.

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1. TOPOLOGICAL CRITERIA

We consider the C- and D-relations of Lazard on a topological space X. The *D-relation* on X is the equivalence relation generated by the relation $x \in \overline{\{y\}}$ $(x, y \in X)$, where $\overline{W}$ denotes the closure of a subset W of X. We put the quotient topology on the set X/D of D-equivalence classes of X. The weak topology on X

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