ARITHMETICALLY EMBEDDABLE LOCAL NOETHER LATTICES

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In [1], Anderson showed that each distributive local Noether lattice can be embedded in the lattice of ideals of a Noetherian ring. However, it is of interest to know when such an embedding can be made into the lattice of ideals of an arithmetical Noetherian ring (that is, a Noetherian ring with a distributive lattice of ideals), and the results of [1] shed no light on this question.

We call a Noether lattice *independently generated* if it satisfies the identity $A(B \wedge C) = AB \wedge AC$, and *basis-annihilating* if it satisfies the identity $(A \vee B)(A \wedge B) = AB$. (Theorems 3 and 9 justify the seemingly strange terminology.) Since the ideals of every arithmetical ring satisfy the stated identities, it is clear that every Noether lattice that is embeddable in the lattice of ideals of an arithmetical ring is both basis-independent and basis-annihilating.

In this paper, we study basis-annihilating and basis-independent Noether lattices, our primary goal being the determination of an answer to the question posed above. However, since the implications of these properties are also interesting outside of the distributive case, we do not limit our considerations to that situation. We note that both properties have received wide attention in rings. (See, for example, [5], [6], [7].)

In [4], K. P. Bogart showed that if E is a principal element in a minimal base of an element A of a distributive local Noether lattice, then E is in every minimal base of A. Since much of our work will be outside of distributive structures, where this obviously need not be the case unless A is principal, it is convenient to introduce the following terminology:

(i) If $\mathscr{L}$ is a local Noether lattice and E is in every minimal base of an element A of $\mathscr{L}$, then E is an *essential generator* of A.

(ii) If $\mathscr{L}$ is local and A has a unique minimal base, then A is *essentially generated.*

(iii) If $\mathscr{L}$ is local and every element is essentially generated, then $\mathscr{L}$ is *essentially generated.*

(iv) If $\mathscr{L}$ is local and no nonprincipal element has an essential generator, then $\mathscr{L}$ is *inessentially generated.*

Finally, we say that a nonlocal Noether lattice is essentially generated or inessentially generated if each of its localizations is. As we have noted, distributive Noether lattices are essentially generated. Lattices of ideals of rings are inessentially generated.

It is easily seen that every independently generated Noether lattice is basis-annihilating (the proof is the same as that given in [6] for rings). And it is fairly easy to see directly that the two conditions are equivalent for Noetherian rings.

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