NAKANO'S THEOREM REVISITED

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This paper is a complement of the authors' paper [2]. In that paper we have provided a new proof of the following theorem (due to H. Nakano): *If* (L, τ) *is a Dedekind complete Riesz space with the Fatou property, then the order intervals of* L *are* τ*-complete.* (For a discussion and the history of this result, the reader is referred to [3], [5], [6], [7], and to the references of [2].) In the course of our proof of Nakano's theorem we made use of a nonelementary result. In this note we show, however, that it is possible to modify the proof of Nakano's theorem as it was presented in [2] so that it becomes elementary. On the other hand, an alternate proof of the same result will also be given.

For notation and basic terminology concerning Riesz spaces we refer the reader to [4]. A *locally solid Riesz space* (L, τ) is a Riesz space L equipped with a locally solid topology τ; that is, equipped with a linear topology τ which has a basis for zero consisting of solid sets. (A subset V of L is said to be a *solid set* if $|u| \leq |v|$ and $v \in V$ implies $u \in V$.) A net $\{u_\alpha\}$ of a Riesz space L *order converges to* u in L, denoted by $u_\alpha \xrightarrow{(o)} u$, if there exists a net $\{v_\alpha\}$ of L (with the same indexing set) such that $|u_\alpha - u| \leq v_\alpha \downarrow \theta$ holds in L. A subset V of a Riesz space is said to be *order closed* if $\{u_\alpha\} \subset V$ and $u_\alpha \xrightarrow{(o)} u$ implies $u \in V$, and V is said to have the *Fatou property* if V is solid and order closed. Note that a solid subset V of a Riesz space L is order closed if and only if $\theta \leq u_\alpha \uparrow u$ in L and $\{u_\alpha\} \subset V$ implies $u \in V$ (see [2, p. 25]).

A locally solid Riesz space (L, τ) has the Fatou property if τ has a basis for zero consisting of sets with the Fatou property. The *topological completion* ($\hat{L}$, $\hat{\tau}$) of a Hausdorff locally solid Riesz space (L, τ) equipped with the cone formed by the closure of L^+ in $\hat{L}$ is a locally solid Riesz space containing L as a Riesz subspace (see [1, Theorem 2.1, p. 109]).

A Riesz subspace L of a Riesz space K is said to be *order dense* in K if $\sup\{v \in L: \theta \leq v \leq u\} = u$ holds in K for all $u \in K^+$. If K is Archimedean this is equivalent to the property that for each $\theta < u \in K$ ($\theta < u$ means, of course, $\theta \leq u$ and $u \neq \theta$), there exists $v \in L$ with $\theta < v \leq u$. In particular, it follows that if L is order dense in K, the embedding of L into K preserves arbitrary suprema and infima.

We continue with a simple but very useful result.

LEMMA. *Assume that* L *is an order dense Riesz subspace of a Riesz space* K. *If* L *is a Dedekind complete Riesz space, then* L *is an ideal of* K.

Proof. Assume $\theta \leq u \leq v$ with $v \in L$ and $u \in K$. Pick a net $\{u_\alpha\} \subset L^+$ with $\theta \leq u_\alpha \uparrow u$ in K and notice that since L is Dedekind complete, $u_\alpha \uparrow w$ holds in L for some $w \in L^+$. But since L is order dense in K, $u_\alpha \uparrow w$ holds also in K. Hence $u = w \in L$; L is an ideal of K.

Received November 14, 1975.

Michigan Math. J. 23 (1976).